ELEMENTARY ILLUSTRATIONS
OF THE
DIFFERENTIAL AND INTEGRAL CALCULUS

DE MORGAN

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BY

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EDITOR'S PREFACE.

THE publication of the present reprint of De Morgan's *Elementary Illustrations of the Differential and Integral Calculus* forms, quite independently of its interest to professional students of mathematics, an integral portion of the general educational plan which the Open Court Publishing Company has been systematically pursuing since its inception,—which is the dissemination among the public at large of sound views of science and of an adequate and correct appreciation of the methods by which truth generally is reached. Of these methods, mathematics, by its simplicity, has always formed the type and ideal, and it is nothing less than imperative that its ways of procedure, both in the discovery of new truth and in the demonstration of the necessity and universality of old truth, should be laid at the foundation of every philosophical education. The greatest achievements in the history of thought—Plato, Descartes, Kant—are associated with the recognition of this principle.

But it is precisely mathematics, and the pure sciences generally, from which the general educated public and independent students have been debarred, and into which they have only rarely attained more than a very meagre insight. The reason of this is twofold. In the first place, the ascendant and consecutive character of mathematical knowledge renders its results absolutely unsusceptible of presentation to persons who are unacquainted with what has gone before, and so necessitates on the part of its devotees a thorough and patient exploration of the field from the very beginning, as distinguished from those sciences which may, so to speak, be begun at the end, and which are consequently cultivated with the greatest zeal. The second reason is that, partly through the exigencies of academic instruction, but mainly through the martinet traditions of antiquity and the influence of mediæval

logic-mongers, the great bulk of the elementary text-books of mathematics have unconsciously assumed a very repellent form,—something similar to what is termed in the theory of protective mimicry in biology "the terrifying form." And it is mainly to this formidableness and touch-me-not character of exterior, concealing withal a harmless body, that the undue neglect of typical mathematical studies is to be attributed.

To this class of books the present work forms a notable exception. It was originally issued as numbers 135 and 140 of the Library of Useful Knowledge (1832), and is usually bound up with De Morgan's large *Treatise on the Differential and Integral Calculus* (1842). Its style is fluent and familiar; the treatment continuous and undogmatic. The main difficulties which encompass the early study of the Calculus are analysed and discussed in connexion with practical and historical illustrations which in point of simplicity and clearness leave little to be desired. No one who will read the book through, pencil in hand, will rise from its perusal without a clear perception of the aim and the simpler fundamental principles of the Calculus, or without finding that the profounder study of the science in the more advanced and more methodical treatises has been greatly facilitated.

The book has been reprinted substantially as it stood in its original form; but the typography has been greatly improved, and in order to render the subject-matter more synoptic in form and more capable of survey, the text has been re-paragraphed and a great number of descriptive sub-headings have been introduced, a list of which will be found in the Contents of the book. An index also has been added.

Persons desirous of continuing their studies in this branch of mathematics, will find at the end of the text a bibliography of the principal English, French, and German works on the subject, as well as of the main Collections of Examples. From the information there given, they may be able to select what will suit their special needs.

THOMAS J. McCORMACK.

LA SALLE, Ill., August, 1899.

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DIFFERENTIAL AND INTEGRAL CALCULUS.

ELEMENTARY ILLUSTRATIONS.

THE Differential and Integral Calculus, or, as it was formerly called in this country [England], the Doctrine of Fluxions, has always been supposed to present remarkable obstacles to the beginner. It is matter of common observation, that any one who commences this study, even with the best elementary works, finds himself in the dark as to the real meaning of the processes which he learns, until, at a certain stage of his progress, depending upon his capacity, some accidental combination of his own ideas throws light upon the subject. The reason of this may be, that it is usual to introduce him at the same time to new principles, processes, and symbols, thus preventing his attention from being exclusively directed to one new thing at a time. It is our belief that this should be avoided; and we propose, therefore, to try the experiment, whether by undertaking the solution of some problems by common algebraical methods, without calling for the reception of more than one new symbol at once, or lessening the immediate evidence of each investigation by reference to general rules, the study of more methodical treatises may not be some-

what facilitated. We would not, nevertheless, that the student should imagine we can remove all obstacles; we must introduce notions, the consideration of which has not hitherto occupied his mind; and shall therefore consider our object as gained, if we can succeed in so placing the subject before him, that two independent difficulties shall never occupy his mind at once.

ON THE RATIO OR PROPORTION OF TWO MAGNITUDES.

The ratio or proportion of two magnitudes is best conceived by expressing them in numbers of some unit when they are commensurable; or, when this is not the case, the same may still be done as nearly as we please by means of numbers. Thus, the ratio of the diagonal of a square to its side is that of $\sqrt{2}$ to 1, which is very nearly that of 14142 to 10000, and is certainly between this and that of 14143 to 10000. Again, any ratio, whatever numbers express it, may be the ratio of two magnitudes, each of which is as small as we please; by which we mean, that if we take any given magnitude, however small, such as the line A, we may find two other lines B and C, each less than A, whose ratio shall be whatever we please. Let the given ratio be that of the numbers m and n. Then, P being a line, mP and nP are in the proportion of m to n; and it is evident, that let m, n, and A be what they may, P can be so taken that mP shall be less than A. This is only saying that P can be taken less than the m^{th} part of A, which is obvious, since A, however small it may be, has its tenth, its hundredth, its thousandth part, etc., as certainly as if it were larger. We are not, therefore, entitled to say that because two magnitudes are diminished, their ratio is

diminished; it is possible that B, which we will suppose to be at first a hundredth part of C, may, after a diminution of both, be its tenth or thousandth, or may still remain its hundredth, as the following example will show:

$$\begin{array}{llll}
C \quad 3600 & 1800 & 36 & 90 \\
B \quad 36 & 1\frac{8}{10} & \frac{36}{100} & 9 \\
B=\frac{1}{100}C & B=\frac{1}{1000}C & B=\frac{1}{100}C & B=\frac{1}{10}C.
\end{array}$$

Here the values of B and C in the second, third, and fourth column are less than those in the first; nevertheless, the ratio of B to C is less in the second column than it was in the first, remains the same in the third, and is greater in the fourth.

In estimating the approach to, or departure from equality, which two magnitudes undergo in consequence of a change in their values, we must not look at their differences, but at the proportions which those differences bear to the whole magnitudes. For example, if a geometrical figure, two of whose sides are 3 and 4 inches now, be altered in dimensions, so that the corresponding sides are 100 and 101 inches, they are nearer to equality in the second case than in the first; because, though the difference is the same in both, namely one inch, it is one third of the least side in the first case, and only one hundredth in the second. This corresponds to the common usage, which rejects quantities, not merely because they are small, but because they are small in proportion to those of which they are considered as parts. Thus, twenty miles would be a material error in talking of a day's journey, but would not be considered worth mentioning in one of three months, and would be called to-

tally insensible in stating the distance between the earth and sun. More generally, if in the two quantities x and $x+a$, an increase of m be given to x, the two resulting quantities $x+m$ and $x+m+a$ are nearer to equality as to their *ratio* than x and $x+a$, though they continue the same as to their *difference;* for $\frac{x+a}{x}=1+\frac{a}{x}$ and $\frac{x+m+a}{x+m}=1+\frac{a}{x+m}$ of which $\frac{a}{x+m}$ is less than $\frac{a}{x}$, and therefore $1+\frac{a}{x+m}$ is nearer to unity than $1+\frac{a}{x}$. In future, when we talk of an approach towards equality, we mean that the ratio is made more nearly equal to unity, not that the difference is more nearly equal to nothing. The second may follow from the first, but not necessarily; still less does the first follow from the second.

ON THE RATIO OF MAGNITUDES THAT VANISH TOGETHER.

It is conceivable that two magnitudes should decrease simultaneously,* so as to vanish or become nothing, together. For example, let a point A move on a circle towards a fixed point B. The arc AB will then diminish, as also the chord AB, and by bringing the point A sufficiently near to B, we may obtain an arc and its chord, both of which shall be smaller than a given line, however small this last may be. But while the magnitudes diminish, we may not assume either that their ratio increases, diminishes, or remains the same, for we have shown that a diminution of two magnitudes is consistent with either of these.

* In introducing the notion of time, we consult only simplicity. It would do equally well to write any number of successive values of the two quantities, and place them in two columns.

We must, therefore, look to each particular case for the change, if any, which is made in the ratio by the diminution of its terms.

Now two suppositions are possible in every increase or diminution of the ratio, as follows: Let M and N be two quantities which we suppose in a state of decrease. The first possible case is that the ratio of M to N may decrease without limit, that is, M may be a smaller fraction of N after a decrease than it was before, and a still smaller after a further decrease, and so on; in such a way, that there is no fraction so small, to which $\frac{M}{N}$ shall not be equal or inferior, if the decrease of M and N be carried sufficiently far. As an instance, form two sets of numbers as in the adjoining table:

M	1	$\frac{1}{20}$	$\frac{1}{400}$	$\frac{1}{8000}$	$\frac{1}{160000}$	etc.
N	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	etc.
Ratio of M to N	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$	$\frac{1}{10000}$	etc.

Here both M and N decrease at every step, but M loses at each step a larger fraction of itself than N, and their ratio continually diminishes. To show that this decrease is without limit, observe that M is at first equal to N, next it is one tenth, then one hundredth, then one thousandth of N, and so on; by continuing the values of M and N according to the same law, we should arrive at a value of M which is a smaller part of N than any which we choose to name; for example, ·000003. The second value of M beyond our table is only one millionth of the corresponding value of N; the ratio is therefore expressed by ·000001

which is less than ·000003. In the same law of formation, the ratio of N to M is also *increased* without limit.

The second possible case is that in which the ratio of M to N, though it increases or decreases, does not increase or decrease without limit, that is, continually approaches to some ratio, which it never will exactly reach, however far the diminution of M and N may be carried. The following is an example:

$$\begin{array}{lccccccc} \text{M} & 1 & \frac{1}{3} & \frac{1}{6} & \frac{1}{10} & \frac{1}{15} & \frac{1}{21} & \frac{1}{28}\text{ etc.} \\ \text{N} & 1 & \frac{1}{4} & \frac{1}{9} & \frac{1}{16} & \frac{1}{25} & \frac{1}{36} & \frac{1}{49}\text{ etc.} \\ \text{Ratio of M to N} & 1 & \frac{4}{3} & \frac{9}{6} & \frac{16}{10} & \frac{25}{15} & \frac{36}{21} & \frac{49}{28}\text{ etc.} \end{array}$$

The ratio here increases at each step, for $\frac{4}{3}$ is greater than 1, $\frac{9}{6}$ than $\frac{4}{3}$, and so on. The difference between this case and the last is, that the ratio of M to N, though perpetually increasing, does not increase without limit; it is never so great as 2, though it may be brought as near to 2 as we please.

To show this, observe that in the successive values of M, the denominator of the second is $1+2$, that of the third $1+2+3$, and so on; whence the denominator of the x^{th} value of M is

$$1+2+3+\ldots\ldots+x, \text{ or } \frac{x(x+1)}{2}$$

Therefore the x^{th} value of M is $\frac{2}{x(x+1)}$, and it is evident that the x^{th} value of N is $\frac{1}{x^2}$, which gives the x^{th} value of the ratio $\frac{\text{M}}{\text{N}} = \frac{2x^2}{x(x+1)}$, or $\frac{2x}{x+1}$, or

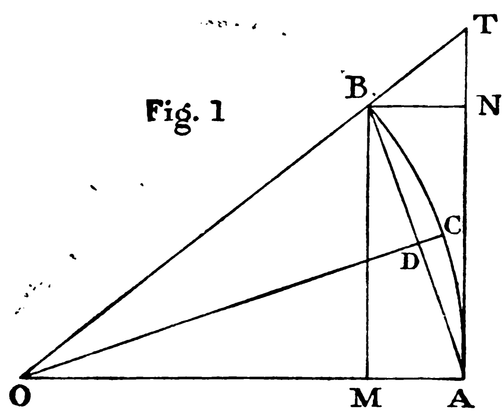
$\frac{x}{x+1} \times 2$. If x be made sufficiently great, $\frac{x}{x+1}$ may be brought as near as we please to 1, since, being $1 - \frac{1}{x+1}$, it differs from 1 by $\frac{1}{x+1}$, which may be made as small as we please. But as $\frac{x}{x+1}$, however great x may be, is always less than 1, $\frac{2x}{x+1}$ is always less than 2. Therefore (1) $\frac{M}{N}$ continually increases; (2) may be brought as near to 2 as we please; (3) can never be greater than 2. This is what we mean by saying that $\frac{M}{N}$ is an increasing ratio, the limit of which is 2. Similarly of $\frac{N}{M}$, which is the reciprocal of $\frac{M}{N}$, we may show (1) that it continually decreases; (2) that it can be brought as near as we please to $\frac{1}{2}$; (3) that it can never be less than $\frac{1}{2}$. This we express by saying that $\frac{N}{M}$ is a decreasing ratio, whose limit is $\frac{1}{2}$.

ON THE RATIOS OF CONTINUOUSLY INCREASING OR DECREASING QUANTITIES.

To the fractions here introduced, there are intermediate fractions, which we have not considered. Thus, in the last instance, M passed from 1 to $\frac{1}{3}$ without any intermediate change. In geometry and mechanics, it is necessary to consider quantities as increasing or decreasing *continuously;* that is, a magnitude does not pass from one value to another without passing through every intermediate value. Thus if one point move towards another on a circle, both the arc and its chord decrease continuously. Let AB (Fig. 1) be an arc of a circle, the centre of which is

O. Let A remain fixed, but let B, and with it the radius OB, move towards A, the point B always remaining on the circle. At every position of B, suppose the following figure. Draw AT touching the circle at A, produce OB to meet AT in T, draw BM and BN perpendicular and parallel to OA, and join BA. Bisect the arc AB in C, and draw OC meeting the chord in D and bisecting it. The right-angled triangles ODA and BMA having a common angle, and also right angles, are similar, as are also BOM and TBN. If now we suppose B to move towards A, before B

Fig. 1



reaches A, we shall have the following results: The arc and chord BA, the lines BM, MA, BT, TN, the angles BOA, COA, MBA, and TBN, will diminish without limit; that is, assign a line and an angle, however small, B can be placed so near to A that the lines and angles above alluded to shall be severally less than the assigned line and angle. Again, OT diminishes and OM increases, but neither without limit, for the first is never less, nor the second greater, than the radius. The angles OBM, MAB, and BTN, increase, but not without limit, each being always less than the right angle, but capable of being made as

near to it as we please, by bringing B sufficiently near to A.

So much for the magnitudes which compose the figure: we proceed to consider their ratios, premising that the arc AB is greater than the chord AB, and less than BN + NA. The triangle BMA being always similar to ODA, their sides change always in the same proportion; and the sides of the first decrease without limit, which is the case with only one side of the second. And since OA and OD differ by DC, which diminishes without limit as compared with OA, the ratio OD $\div$ OA is an increasing ratio whose limit is 1. But OD $\div$ OA $=$ BM $\div$ BA. We can therefore bring B so near to A that BM and BA shall differ by as small a fraction of either of them as we please.

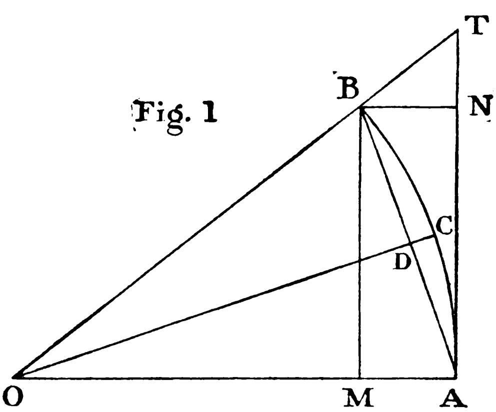
To illustrate this result from the trigonometrical tables, observe that if the radius OA be the linear unit, and $\angle$ BOA $= \theta$, BM and BA are respectively $\sin\theta$ and $2\sin\frac{1}{2}\theta$. Let $\theta = 1°$; then $\sin\theta = \cdot 0174524$ and $2\sin\frac{1}{2}\theta = \cdot 0174530$; whence $2\sin\frac{1}{2}\theta \div \sin\theta =$ $1 \cdot 00003$ very nearly, so that BM differs from BA by less than four of its own hundred-thousandth parts. If $\angle$ BOA $= 4'$, the same ratio is $1 \cdot 0000002$, differing from unity by less than the hundredth part of the difference in the last example.

Again, since DA diminishes continually and without limit, which is not the case either with OD or OA, the ratios OD $\div$ DA and OA $\div$ DA increase without limit. These are respectively equal to BM $\div$ MA and BA $\div$ MA; whence it appears that, let a number be ever so great, B can be brought so near to A, that BM and BA shall each contain MA more times than there are units in that number. Thus if $\angle$ BOA $= 1°$, BM $\div$ MA $= 114 \cdot 589$ and BA $\div$ MA $= 114 \cdot 593$ very

nearly; that is, BM and BA both contain MA more than 114 times. If $\angle$BOA$=4'$, BM $\div$ MA$=$ 1718·8732, and BA $\div$ MA$=$1718·8375 very nearly; or BM and BA both contain MA more than 1718 times.

No difficulty can arise in conceiving this result, if the student recollect that the degree of greatness or smallness of two magnitudes determines nothing as to their ratio; since every quantity N, however small, can be divided into as many parts as we please, and has therefore another small quantity which is its mil-

Fig. 1



lionth or hundred-millionth part, as certainly as if it had been greater. There is another instance in the line TN, which, since TBN is similar to BOM, decreases continually with respect to TB, in the same manner as does BM with respect to OB.

The arc BA always lies between BA and BN$+$NA, or BM$+$MA; hence $\dfrac{\text{arc BA}}{\text{chord BA}}$ lies between 1 and $\dfrac{\text{BM}}{\text{BA}}+\dfrac{\text{MA}}{\text{BA}}$. But $\dfrac{\text{BM}}{\text{BA}}$ has been shown to approach continually towards 1, and $\dfrac{\text{MA}}{\text{BA}}$ to decrease without limit; hence $\dfrac{\text{arc BA}}{\text{chord BA}}$ continually approaches towards

1. If $\angle \mathrm{BOA}=1^\circ$, $\dfrac{\text{arc BA}}{\text{chord BA}}=\cdot 0174533 \div \cdot 0174530=$ $1\cdot 00002$, very nearly. If $\angle \mathrm{BOA}=4'$, it is less than $1\cdot 0000001$.

We now proceed to illustrate the various phrases which have been used in enunciating these and similar propositions.

THE NOTION OF INFINITELY SMALL QUANTITIES.

It appears that it is possible for two quantities m and $m+n$ to decrease together in such a way, that n continually decreases with respect to m, that is, becomes a less and less part of m, so that $\frac{n}{m}$ also decreases when n and m decrease. Leibnitz,* in introducing the Differential Calculus, presumed that in such a case, n might be taken so small as to be utterly inconsiderable when compared with m, so that $m+n$ might be put for m, or *vice versa*, without any error at all. In this case he used the phrase that n is *infinitely* small with respect to m.

The following example will illustrate this term. Since $(a+h)^2=a^2+2ah+h^2$, it appears that if a be increased by h, a^2 is increased by $2ah+h^2$. But if h be taken very small, h^2 is very small with respect to h, for since $1:h::h:h^2$, as many times as 1 contains h, so many times does h contain h^2; so that by taking

* Leibnitz was a native of Leipsic, and died in 1716, aged 70. His dispute with Newton, or rather with the English mathematicians in general, about the invention of Fluxions, and the virulence with which it was carried on, are well known. The decision of modern times appears to be that both Newton and Leibnitz were independent inventors of this method. It has, perhaps, not been sufficiently remarked how nearly several of their predecessors approached the same ground; and it is a question worthy of discussion, whether either Newton or Leibnitz might not have found broader hints in writings accessible to both, than the latter was ever asserted to have received from the former.

h sufficiently small, h may be made to be as many times h^2 as we please. Hence, in the words of Leibnitz, if h be taken *infinitely* small, h^2 is *infinitely* small with respect to h, and therefore $2ah+h^2$ is the same as $2ah$; or if a be increased by an infinitely small quantity h, a^2 is increased by another infinitely small quantity $2ah$, which is to h in the proportion of $2a$ to 1.

In this reasoning there is evidently an absolute error; for it is impossible that h can be so small, that $2ah+h^2$ and $2ah$ shall be the same. The word *small* itself has no precise meaning; though the word *smaller*, or *less*, as applied in comparing one of two magnitudes with another, is perfectly intelligible. Nothing is either small or great in itself, these terms only implying a relation to some other magnitude of the same kind, and even then varying their meaning with the subject in talking of which the magnitude occurs, so that both terms may be applied to the same magnitude: thus a large field is a very small part of the earth. Even in such cases there is no natural point at which smallness or greatness commences. The thousandth part of an inch may be called a small distance, a mile moderate, and a thousand leagues great, but no one can fix, even for himself, the precise mean between any of these two, at which the one quality ceases and the other begins. These terms are not therefore a fit subject for mathematical discussion, until some more precise sense can be given to them, which shall prevent the danger of carrying away with the words, some of the confusion attending their use in ordinary language. It has been usual to say that when h decreases from any given value towards nothing, h^2 will become *small* as compared with h, because,

let a number be ever so great, h will, before it becomes nothing, contain h^2 more than that number of times. Here all dispute about a standard of smallness is avoided, because, be the standard whatever it may, the proportion of h^2 to h may be brought under it. It is indifferent whether the thousandth, ten-thousandth, or hundred-millionth part of a quantity is to be considered small enough to be rejected by the side of the whole, for let h be $\frac{1}{1000}$, $\frac{1}{10,000}$, or $\frac{1}{100,000,000}$ of the unit, and h will contain h^2, 1000, 10,000, or 100,000,000 of times.

The proposition, therefore, that h can be taken so small that $2ah + h^2$ and $2ah$ are rigorously equal, though not true, and therefore entailing error upon all its subsequent consequences, yet is of this character, that, by taking h sufficiently small, all errors may be made as small as we please. The desire of combining simplicity with the appearance of rigorous demonstration, probably introduced the notion of infinitely small quantities; which was further established by observing that their careful use never led to any error. The method of stating the above-mentioned proposition in strict and rational terms is as follows: If a be increased by h, a^2 is increased by $2ah + h^2$, which, whatever may be the value of h, is to h in the proportion of $2a + h$ to 1. The smaller h is made, the more near does this proportion diminish towards that of $2a$ to 1, to which it may be made to approach within any quantity, if it be allowable to take h as small as we please. Hence the ratio, *increment of* $a^2 \div$ *increment of* a, is a decreasing ratio, whose limit is $2a$.

In further illustration of the language of Leibnitz, we observe, that according to his phraseology, if AB

be an *infinitely* small arc, the chord and arc AB are equal, or the circle is a polygon of an *infinite* number of *infinitely* small rectilinear sides. This should be considered as an abbreviation of the proposition proved (page 10), and of the following: If a polygon be inscribed in a circle, the greater the number of its sides, and the smaller their lengths, the more nearly will the perimeters of the polygon and circle be equal to one another; and further, if any straight line be given, however small, the difference between the perimeters of the polygon and circle may be made less than that line, by sufficient increase of the number of sides and diminution of their lengths. Again, it would be said (Fig. 1) that if AB be infinitely small, MA is infinitely less than BM. What we have proved is, that MA may be made as small a part of BM as we please, by sufficiently diminishing the arc BA.

ON FUNCTIONS.

An algebraical expression which contains x in any way, is called a *function* of x. Such are x^2+a^2, $\frac{a+x}{a-x}$, $\log(x+y)$, $\sin 2x$. An expression may be a function of more quantities than one, but it is usual only to name those quantities of which it is necessary to consider a change in the value. Thus if in x^2+a^2 x only is considered as changing its value, this is called a function of x; if x and a both change, it is called a function of x and a. Quantities which change their values during a process, are called *variables*, and those which remain the same, *constants*; and variables which we change at pleasure are called *independent*, while those whose changes necessarily follow from

the changes of others are called *dependent*. Thus in Fig. 1, the length of the radius OB is a constant, the arc AB is the independent variable, while BM, MA, the chord AB, etc., are dependent. And, as in algebra we reason on numbers by means of general symbols, each of which may afterwards be particularised as standing for any number we please, unless specially prevented by the conditions of the problem, so, in treating of functions, we use general symbols, which may, under the restrictions of the problem, stand for any function whatever. The symbols used are the letters F, f, Φ, φ, ψ; $\varphi(x)$ and $\psi(x)$, or φx and ψx, may represent any functions of x, just as x may represent any number. Here it must be borne in mind that φ and ψ do not represent numbers which multiply x, but are *the abbreviated directions to perform certain operations with x and constant quantities*. Thus, if $\varphi x = x + x^2$, φ is equivalent to a direction to add x to its square, and the whole φx stands for the result of this operation. Thus, in this case, $\varphi(1) = 2$; $\varphi(2) = 6$; $\varphi a = a + a^2$; $\varphi(x+h) = x + h + (x+h)^2$; $\varphi \sin x = \sin x + (\sin x)^2$. It may be easily conceived that this notion is useless, unless there are propositions which are generally true of all functions, and which may be made the foundation of general reasoning.

INFINITE SERIES.

To exercise the student in this notation, we proceed to explain one of these functions which is of most extensive application and is known by the name of *Taylor's Theorem*. If in φx, any function of x, the value of x be increased by h, or $x + h$ be substituted instead of x, the result is denoted by $\varphi(x+h)$. It

will generally* happen that this is either greater or less than φx, and h is called the *increment* of x, and $\varphi(x+h) - \varphi x$ is called the *increment* of φx, which is negative when $\varphi(x+h) < \varphi x$. It may be proved that $\varphi(x+h)$ can generally be expanded in a series of the form

$$\varphi x + ph + qh^2 + rh^3 + \text{etc., } \textit{ad infinitum},$$

which contains none but whole and positive powers of h. It will happen, however, in many functions, that one or more values can be given to x for which it is impossible to expand $f(x+h)$ without introducing negative or fractional powers. These cases are considered by themselves, and the values of x which produce them are called *singular* values.

As the notion of a series which has no end of its terms, may be new to the student, we will now proceed to show that there may be series so constructed, that the addition of any number of their terms, however great, will always give a result less than some determinate quantity. Take the series

$$1 + x + x^2 + x^3 + x^4 + \text{etc.},$$

in which x is supposed to be less than unity. The first two terms of this series may be obtained by dividing $1 - x^2$ by $1 - x$; the first three by dividing $1 - x^3$ by $1 - x$; and the first n terms by dividing $1 - x^n$ by $1 - x$. If x be less than unity, its successive powers decrease without limit;† that is, there is

*This word is used in making assertions which are for the most part true, but admit of exceptions, few in number when compared with the other cases. Thus it generally happens that $x^2 - 10x + 40$ is greater than 15, with the exception only of the case where $x = 5$. It is generally true that a line which meets a circle in a given point meets it again, with the exception only of the tangent.

†This may be proved by means of the proposition established in the *Study of Mathematics* (Chicago: The Open Court Publishing Co., Reprint Edition),

no quantity so small, that a power of x cannot be found which shall be smaller. Hence by taking n sufficiently great, $\frac{1-x^n}{1-x}$ or $\frac{1}{1-x}-\frac{x^n}{1-x}$ may be brought as near to $\frac{1}{1-x}$ as we please, than which, however, it must always be less, since $\frac{x^n}{1-x}$ can never entirely vanish, whatever value n may have, and therefore there is always something subtracted from $\frac{1}{1-x}$. It follows, nevertheless, that $1+x+x^2+$ etc., if we are at liberty to take as many terms as we please, can be brought as near as we please to $\frac{1}{1-x}$, and in this sense we say that

$$\frac{1}{1-x}=1+x+x^2+x^3+\text{ etc., } \textit{ad infinitum}.$$

CONVERGENT AND DIVERGENT SERIES.

A series is said to be *convergent* when the sum of its terms tends towards some limit; that is, when, by taking any number of terms, however great, we shall never exceed some certain quantity. On the other hand, a series is said to be *divergent* when the sum of a number of terms may be made to surpass any quantity, however great. Thus of the two series,

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\text{ etc.}$$

and

$$1+2+4+8+\text{ etc.}$$

the first is convergent, by what has been shown, and the second is evidently divergent. A series cannot be convergent, unless its separate terms decrease, so as,

page 247. For $\frac{m}{n}\times\frac{m}{n}$ is formed (if m be less than n) by dividing $\frac{m}{n}$ into n parts, and taking away $n-m$ of them.

at last, to become less than any given quantity. And the terms of a series may at first increase and afterwards decrease, being apparently divergent for a finite number of terms, and convergent afterwards. It will only be necessary to consider the latter part of the series.

Let the following series consist of terms decreasing without limit:

$$a+b+c+d+\ldots\ldots+k+l+m+\ldots\ldots,$$

which may be put under the form

$$a(1+\frac{b}{a}+\frac{c}{b}\frac{b}{a}+\frac{d}{c}\frac{c}{b}\frac{b}{a}+\text{etc.});$$

the same change of form may be made, beginning from any term of the series, thus:

$$k+l+m+\text{etc.}=k\,(1+\frac{l}{k}+\frac{m}{l}\frac{l}{k}+\text{etc.}).$$

We have introduced the new terms $\frac{b}{a}$, $\frac{c}{b}$, etc., or the ratios which the several terms of the original series bear to those immediately preceding. It may be shown (1) that if the terms of the series $\frac{b}{a}$, $\frac{c}{b}$, $\frac{d}{c}$, etc., come at last to be less than unity, and afterwards either continue to approximate to a limit which is less than unity, or decrease without limit, the series $a+b+c+$ etc., is convergent; (2) if the limit of the terms $\frac{b}{a}$, $\frac{c}{b}$, etc., is either greater than unity, or if they increase without limit, the series is divergent.

(1a). Let $\frac{l}{k}$ be the first which is less than unity, and let the succeeding ratios $\frac{m}{l}$, etc., decrease, either with or without limit, so that $\frac{l}{k}>\frac{m}{l}>\frac{n}{m}$, etc.; whence it follows, that of the two series,

$$k(1 + \frac{l}{k} + \frac{l}{k}\frac{l}{k} + \frac{l}{k}\frac{l}{k}\frac{l}{k} + \text{etc.}),$$

$$k(1 + \frac{l}{k} + \frac{l}{k}\frac{m}{l} + \frac{l}{k}\frac{m}{l}\frac{n}{m} + \text{etc.}),$$

the first is greater than the second. But since $\frac{l}{k}$ is less than unity, the first can never surpass $k \times \frac{1}{1-\frac{l}{k}}$, or $\frac{k^2}{k-l}$, and is convergent; the second is therefore convergent. But the second is no other than $k+l+m+$ etc.; therefore the series $a+b+c+$ etc., is convergent from the term k.

(1 b.) Let $\frac{l}{k}$ be less than unity, and let the successive ratios $\frac{l}{k}, \frac{m}{l}$, etc., increase, never surpassing a limit A, which is less than unity. Hence of the two series,

$$k(1 + \mathrm{A} + \mathrm{A}\,\mathrm{A} + \mathrm{A}\,\mathrm{A}\,\mathrm{A} + \text{etc.}),$$

$$k(1 + \frac{l}{k} + \frac{l}{k}\frac{m}{l} + \frac{l}{k}\frac{m}{l}\frac{n}{m} + \text{etc.}),$$

the first is the greater. But since A is less than unity, the first is convergent; whence, as before, $a+b+c+$ etc., converges from the term k.

(2) The second theorem on the divergence of series we leave to the student's consideration, as it is not immediately connected with our object.

TAYLOR'S THEOREM. DERIVED FUNCTIONS.

We now proceed to the series

$$ph + qh^2 + rh^3 + sh^4 + \text{etc.},$$

in which we are at liberty to suppose h as small as we please. The successive ratios of the terms to those

immediately preceding are $\frac{qh^2}{ph}$ or $\frac{q}{p}h$, $\frac{rh^3}{qh^2}$ or $\frac{r}{q}h$, $\frac{sh^4}{rh^3}$ or $\frac{s}{r}h$, etc. If, then, the terms $\frac{q}{p}$, $\frac{r}{q}$, $\frac{s}{r}$, etc., are always less than a finite limit A, or become so after a definite number of terms, $\frac{q}{p}h$, $\frac{r}{q}h$, etc., will always be, or will at length become, less than Ah. And since h may be what we please, it may be so chosen that Ah shall be less than unity, for which h must be less than $\frac{1}{A}$. In this case, by theorem (1b), the series is convergent; it follows, therefore, that a value of h can always be found so small that $ph+qh^2+rh^3+$etc., shall be convergent, at least unless the coefficients p, q, r, etc., be such that the ratio of any one to the preceding increases without limit, as we take more distant terms of the series. This never happens in the developments which we shall be required to consider in the Differential Calculus.

We now return to $\varphi(x+h)$, which we have asserted (page 16) can be expanded (with the exception of some particular values of x) in a series of the form $\varphi x+ph+qh^2+$etc. The following are some instances of this development derived from the Differential Calculus, most of which are also to be found in treatises on algebra:

$$(x+h)^n = x^n + nx^{n-1}h + n(n-1)x^{n-2}\frac{h^2}{2} + n(n-1)(n-2)x^{n-3}\frac{h^3}{2.3} \text{ etc.}$$

$$a^{x+h} = a^x + ka^x h^* + k^2a^x\frac{h^2}{2} + k^3a^x\frac{h^3}{2.3} \text{ etc.}$$

$$\log(x+h) = \log x + \frac{1}{x}h - \frac{1}{x^2}\frac{h^2}{2} + \frac{2}{x^3}\frac{h^3}{2.3} \text{ etc.}$$

$$\sin(x+h) = \sin x + \cos x\, h - \sin x\frac{h^2}{2}\dagger - \cos x\frac{h^3}{2.3} \text{ etc.}$$

*Here k is the Naperian or hyperbolic logarithm of a; that is, the common logarithm of a divided by .434294482.

† In the last two series the terms are positive and negative in pairs.

$$\cos(x+h) = \cos x - \sin x\, h \qquad - \cos x \frac{h^2}{2} \qquad + \sin x \frac{h^3}{2.3} \text{ etc.}$$

It appears, then, that the development of $\varphi(x+h)$ consists of certain functions of x, the first of which is φx itself, and the remainder of which are multiplied by h, $\frac{h^2}{2}$, $\frac{h^3}{2.3}$, $\frac{h^4}{2.3.4}$, and so on. It is usual to denote the coefficients of these divided powers of h by $\varphi' x$, $\varphi'' x$, $\varphi''' x$,* etc., where φ', φ'', etc., are merely functional symbols, as is φ itself; but it must be recollected that $\varphi' x$, $\varphi'' x$, etc., are rarely, if ever, employed to signify anything except the coefficients of h, $\frac{h^2}{2}$, etc., in the development of $\varphi(x+h)$. Hence this development is usually expressed as follows:

$$\varphi(x+h) = \varphi x + \varphi' x\, h + \varphi'' x \frac{h^2}{2} + \varphi''' x \frac{h^3}{2.3} + \text{etc.}$$

Thus, when $\varphi x = x^n$, $\varphi' x = n x^{n-1}$, $\varphi'' x = n(n-1) x^{n-2}$, etc.; when $\varphi x = \sin x$, $\varphi' x = \cos x$, $\varphi'' x = -\sin x$, etc. In the first case $\varphi'(x+h) = n(x+h)^{n-1}$, $\varphi''(x+h) = n(n-1)(x+h)^{n-2}$; and in the second $\varphi'(x+h) = \cos(x+h)$, $\varphi''(x+h) = -\sin(x+h)$.

The following relation exists between φx, $\varphi' x$, $\varphi'' x$, etc. In the same manner as $\varphi' x$ is the coefficient of h in the development of $\varphi(x+h)$, so $\varphi'' x$ is the coefficient of h in the development of $\varphi'(x+h)$, and $\varphi''' x$ is the coefficient of h in the development of $\varphi''(x+h)$; $\varphi^{iv} x$ is the coefficient of h in the development of $\varphi'''(x+h)$, and so on.

The proof of this is equivalent to *Taylor's Theorem* already alluded to (page 15); and the fact may be verified in the examples already given. When $\varphi x = a^x$, $\varphi' x = k a^x$, and $\varphi'(x+h) = k a^{x+h} = k(a^x + k a^x h + \text{etc.})$. The coefficient of h is here $k^2 a^x$, which is the

* Called *derived functions* or *derivatives.—Ed.*

same as $\varphi''x$. (See the second example of the preceding table.) Again, $\varphi''(x+h)=k^2a^{x+h}=k^2(a^x+k\,a^x h+\text{etc.})$, in which the coefficient of h is $k^3 a^x$, the same as $\varphi'''x$. Again, if $\varphi x=\log x$, $\varphi' x=\dfrac{1}{x}$, and $\varphi'(x+h)=\dfrac{1}{x+h}=\dfrac{1}{x}-\dfrac{h}{x^2}+\text{etc.}$, as appears by common division. Here the coefficient of h is $-\dfrac{1}{x^2}$, which is the same as $\varphi''x$ in the third example. Also $\varphi''(x+h)=-\dfrac{1}{(x+h)^2}=-(x+h)^{-2}$, which by the Binomial Theorem is $-(x^{-2}-2x^{-3}h+\text{etc.})$. The coefficient of h is $2x^{-3}$ or $\dfrac{2}{x^3}$, which is $\varphi'''x$ in the same example.

DIFFERENTIAL COEFFICIENTS.

It appears, then, that if we are able to obtain the coefficient of h in the development of *any* function whatever of $x+h$, we can obtain all the other coefficients, since we can thus deduce $\varphi' x$ from φx, $\varphi'' x$ from $\varphi' x$, and so on. It is usual to call $\varphi' x$ the first differential coefficient of φx, $\varphi'' x$ the second differential coefficient of φx, or the first differential coefficient of $\varphi' x$; $\varphi''' x$ the third differential coefficient of φx, or the second of $\varphi' x$, or the first of $\varphi'' x$; and so on.* The name is derived from a method of obtaining $\varphi' x$, etc., which we now proceed to explain.

Let there be any function of x, which we call φx, in which x is increased by an increment h; the function then becomes

$$\varphi x+\varphi' x\,h+\varphi'' x\,\frac{h^2}{2}+\varphi''' x\,\frac{h^3}{2.3}+\text{etc.}$$

* The first, second, third, etc., differential coefficients, as thus obtained, are also called the first, second, third, etc., *derivatives*.—*Ed.*

The original value φx is increased by the increment

$$\varphi' x \; h + \varphi'' x \frac{h^2}{2} + \varphi''' x \frac{h^3}{2.3} + \text{etc.};$$

whence (h being the increment of x)

$$\frac{\textit{increment of } \varphi x}{\textit{increment of } x} = \varphi' x + \varphi'' x \frac{h}{2} + \varphi''' x \frac{h^2}{2.3} + \text{etc.},$$

which is an expression for the ratio which the increment of a function bears to the increment of its variable. It consists of two parts. The one, $\varphi' x$, into which h does not enter, depends on x only; the remainder is a series, every term of which is multiplied by some power of h, and which therefore diminishes as h diminishes, and may be made as small as we please by making h sufficiently small.

To make this last assertion clear, observe that all the ratio, except its first term $\varphi' x$, may be written as follows:

$$h(\varphi'' x \frac{1}{2} + \varphi''' x \frac{h}{2.3} + \text{etc.});$$

the second factor of which (page 19) is a convergent series whenever h is taken less than $\frac{1}{A}$, where A is the limit towards which we approximate by taking the coefficients $\varphi'' x \times \frac{1}{2}$, $\varphi''' x \times \frac{1}{2.3}$, etc., and forming the ratio of each to the one immediately preceding. This limit, as has been observed, is finite in every series which we have occasion to use; and therefore a value for h can be chosen so small, that for it the series in the last-named formula is convergent; still more will it be so for every smaller value of h. Let the series be called P. If P be a finite quantity, which decreases when h decreases, Ph can be made as small as we please by sufficiently diminishing

h; whence $\varphi'x+\mathrm{P}h$ can be brought as near as we please to $\varphi'x$. Hence the ratio of the increments of φx and x, produced by changing x into $x+h$, though never equal to $\varphi'x$, approaches towards it as h is diminished, and may be brought as near as we please to it, by sufficiently diminishing h. Therefore to find the coefficient of h in the development of $\varphi(x+h)$, find $\varphi(x+h)-\varphi x$, divide it by h, and find the limit towards which it tends as h is diminished.

In any series such as

$$a+bh+ch^2\ldots\ldots+kh^n+lh^{n+1}+mh^{n+2}+\text{etc.}$$

which is such that some given value of h will make it convergent, it may be shown that h can be taken so small that any one term shall contain all the succeeding ones as often as we please. Take any one term, as kh^n. It is evident that, be h what it may,

$$kh^n : lh^{n+1}+mh^{n+2}+\text{etc.},\ :: \ k : lh+mh^2+\text{etc.},$$

the last term of which is $h(l+mh+\text{etc.})$. By reasoning similar to that in the last paragraph, we can show that this may be made as small as we please, since one factor is a series which is always finite when h is less than $\frac{1}{\mathrm{A}}$, and the other factor h can be made as small as we please. Hence, since k is a given quantity, independent of h, and which therefore remains the same during all the changes of h, the series $h(l+mh+\text{etc.})$ can be made as small a part of k as we please, since the first diminishes without limit, and the second remains the same. By the proportion above established, it follows then that $lh^{n+1}+mh^{n+2}+\text{etc.}$, can be made as small a part as we please of kh^n. It follows, therefore, that if, instead of the full development of $\varphi(x+h)$, we use only its two first

terms $\varphi x + \varphi' x h$, the error thereby introduced may, by taking h sufficiently small, be made as small a portion as we please of the small term $\varphi' x h$.

THE NOTATION OF THE DIFFERENTIAL CALCULUS.

The first step usually made in the Differential Calculus is the determination of $\varphi' x$ for all possible values of φx, and the construction of general rules for that purpose. Without entering into these we proceed to explain the notation which is used, and to apply the principles already established to the solution of some of those problems which are the peculiar province of the Differential Calculus.

When any quantity is increased by an increment, which, consistently with the conditions of the problem, may be supposed as small as we please, this increment is denoted, not by a separate letter, but by prefixing the letter d, either followed by a full stop or not, to that already used to signify the quantity. For example, the increment of x is denoted under these circumstances by dx; that of φx by $d.\varphi x$; that of x^n by $d.x^n$. If instead of an increment a decrement be used, the sign of dx, etc., must be changed in all expressions which have been obtained on the supposition of an increment; and if an increment obtained by calculation proves to be negative, it is a sign that a quantity which we imagined was increased by our previous changes, was in fact diminished. Thus, if x becomes $x + dx$, x^2 becomes $x^2 + d.x^2$. But this is also $(x + dx)^2$ or $x^2 + 2x\ dx + (dx)^2$; whence $d.x^2 = 2x\ dx + (dx)^2$. Care must be taken not to confound $d.x^2$, the increment of x^2, with $(dx)^2$, or, as it is often written, dx^2, the square of the increment of x. Again,

if x becomes $x+dx$, $\frac{1}{x}$ becomes $\frac{1}{x}+d.\frac{1}{x}$ and the change of $\frac{1}{x}$ is $\frac{1}{x+dx}-\frac{1}{x}$ or $-\frac{dx}{x^2+x\,dx}$; showing that an increment of x produces a decrement in $\frac{1}{x}$.

It must not be imagined that because x occurs in the symbol dx, the value of the latter in any way depends upon that of the former: both the first value of x, and the quantity by which it is made to differ from its first value, are at our pleasure, and the letter d must merely be regarded as an abbreviation of the words "*difference of.*" In the first example, if we divide both sides of the resulting equation by dx, we have $\frac{d.x^2}{dx}=2x+dx$. The smaller dx is supposed to be, the more nearly will this equation assume the form $\frac{d.x^2}{dx}=2x$, and the ratio of $2x$ to 1 is the limit of the ratio of the increment of x^2 to that of x; to which this ratio may be made to approximate as nearly as we please, but which it can never actually reach. In the Differential Calculus, the limit of the ratio only is retained, to the exclusion of the rest, which may be explained in either of the two following ways:

(1) The fraction $\frac{d.x^2}{dx}$ may be considered as standing, not for any value which it can actually have as long as dx has a real value, but for the limit of all those values which it assumes while dx diminishes. In this sense the equation $\frac{d.x^2}{dx}=2x$ is strictly true. But here it must be observed that the algebraical meaning of the sign of division is altered, in such a way that it is no longer allowable to use the numerator and denominator separately, or even at all to con-

sider them as quantities. If $\frac{dy}{dx}$ stands, not for the ratio of two quantities, but for the limit of that ratio, which cannot be obtained by taking any real value of dx, however small, the whole $\frac{dy}{dx}$ may, by convention, have a meaning, but the separate parts dy and dx have none, and can no more be considered as separate quantities whose ratio is $\frac{dy}{dx}$, than the two loops of the figure 8 can be considered as separate numbers whose sum is eight. This would be productive of no great inconvenience if it were never required to separate the two; but since all books on the Differential Calculus and its applications are full of examples in which deductions equivalent to assuming $dy = 2x\,dx$ are drawn from such an equation as $\frac{dy}{dx} = 2x$, it becomes necessary that the first should be explained, independently of the meaning first given to the second. It may be said, indeed, that if $y = x^2$, it follows that $\frac{dy}{dx} = 2x + dx$, in which, *if we make* $dx = 0$, the result is $\frac{dy}{dx} = 2x$. But if $dx = 0$, dy also $= 0$, and this equation should be written $\frac{0}{0} = 2x$, as is actually done in some treatises on the Differential Calculus,* to the great confusion of the learner. Passing over the difficulties† of the fraction $\frac{0}{0}$, still the former objection recurs, that the equation $dy = 2x\,dx$ cannot be used

*This practice was far more common in the early part of the century than now, and was due to the precedent of Euler (1755). For the sense in which Euler's view was correct, see the *Encyclopædia Britannica*, art. *Infinitesimal Calculus*, Vol. XII, p. 14, 2nd column.—*Ed.*

†See *Study of Mathematics* (Reprint Edition, Chicago: The Open Court Publishing Co., 1898), page 126.

(and it *is* used even by those who adopt this explanation) without supposing that 0, which merely implies an absence of all magnitude, can be used in different senses, so that one 0 may be contained in another a certain number of times. This, even if it can be considered as intelligible, is a notion of much too refined a nature for a beginner.

(2) The presence of the letter *d* is an indication, not only of an increment, but of an increment which we are at liberty to suppose as small as we please. The processes of the Differential Calculus are intended to deduce relations, not between the ratios of different increments, but between the limits to which those ratios approximate, when the increments are decreased. And it may be true of some parts of an equation, that though the taking of them away would alter the relation between dy and dx, it would not alter the limit towards which their ratio approximates, when dx and dy are diminished. For example, $dy = 2x\,dx + (dx)^2$. If $x = 4$ and $dx = \cdot 01$, then $dy = \cdot 0801$ and $\frac{dy}{dx} = 8 \cdot 01$. If $dx = \cdot 0001$, $dy = \cdot 00080001$ and $\frac{dy}{dx} = 8 \cdot 0001$. The limit of this ratio, to which we shall come still nearer by making dx still smaller, is 8. The term $(dx)^2$, though its presence affects the value of dy and the ratio $\frac{dy}{dx}$, does not affect the limit of the latter, for in $\frac{dy}{dx}$ or $2x + dx$, the latter term dx, which arose from the term $(dx)^2$, diminishes continually and without limit. If, then, we throw away the term $(dx)^2$, the consequence is that, make dx what we may, we never obtain dy as it would be if correctly deduced from the equation $y = x^2$, but we obtain the limit of the ratio of dy to dx. If we throw away all powers of

dx above the first, and use the equations so obtained, all ratios formed from these last, or their consequences, are themselves the limiting ratios of which we are in search. *The equations which we thus use are not absolutely true in any case, but may be brought as near as we please to the truth,* by making dy and dx sufficiently small. If the student at first, instead of using $dy = 2x\,dx$, were to write it thus, $dy = 2x\,dx +$ etc., the *etc.* would remind him that there are other terms; *necessary,* if the value of dy corresponding to any value of dx is to be obtained; *unnecessary,* if the *limit* of the ratio of dy to dx is all that is required.

We must adopt the first of these explanations when dy and dx appear in a fraction, and the second when they are on opposite sides of an equation.

ALGEBRAICAL GEOMETRY.

If two straight lines be drawn at right angles to each other, dividing the whole of their plane into four parts, one lying in each right angle, the situation of any point is determined when we know, (1) in which angle it lies, and (2) its perpendicular distances from the two right lines. Thus (Fig. 2) the point P lying in the angle AOB, is known when PM and PN, or when OM and PM are known; for, though there is an infinite number of points whose distance from OA only is the same as that of P, and an infinite number of others, whose distance from OB is the same as that of P, there is no other point whose distances from both lines are the same as those of P. The line OA is called the axis of x, because it is usual to denote any variable distance measured on or parallel to OA by the letter x. For a similar reason, OB is called

the axis of y. The *co-ordinates** or perpendicular distances of a point P which is supposed to vary its position, are thus denoted by x and y; hence OM or PN is x, and PM or ON is y. Let a linear unit be chosen, so that any number may be represented by a straight line. Let the point M, setting out from O, move in the direction OA, always carrying with it the indefinitely extended line MP perpendicular to OA. While this goes on, let P move upon the line MP in such a way, that MP or y is always equal to a given function of OM or x; for example, let $y=x^2$, or let the num-

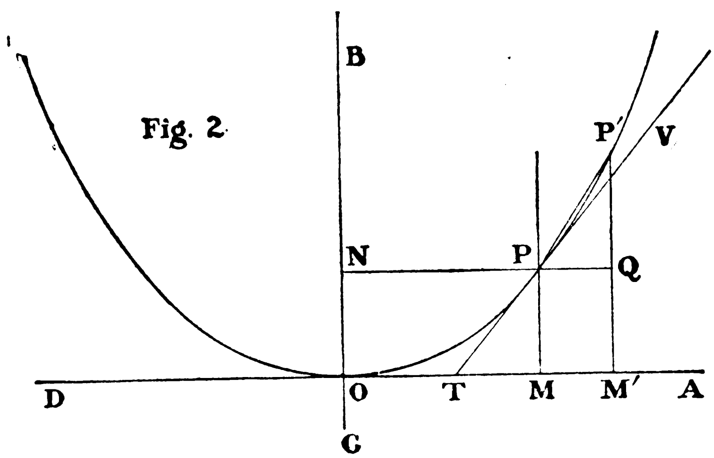


Fig. 2.

ber of units in PM be the square of the number of units in OM. As O moves towards A, the point P will, by its motion on MP, compounded with the motion of the line MP itself, describe a curve OP, in which PM is less than, equal to, or greater than, OM, according as OM is less than, equal to, or greater than the linear unit. It only remains to show how the other branch of this curve is deduced from the equation $y=x^2$. And to this end we shall first have to interpolate a few remarks.

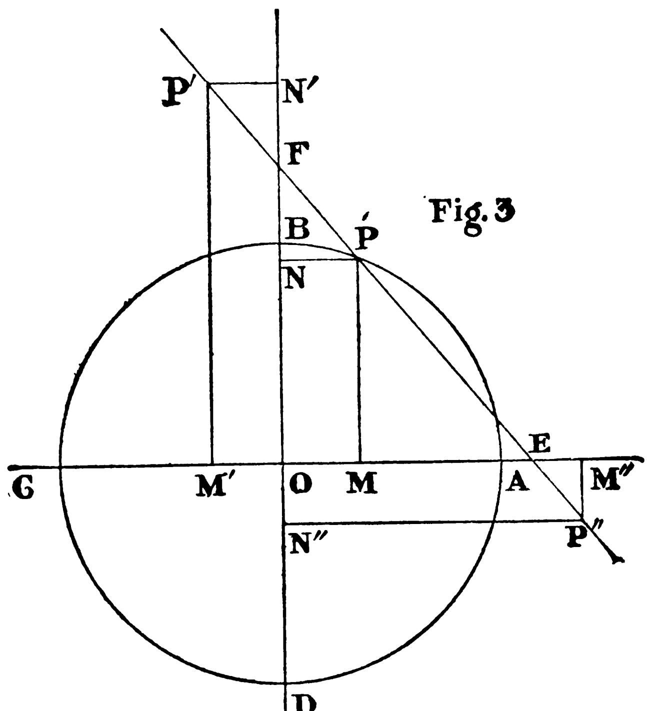
* The distances OM and MP are called the *co-ordinates* of the point P. It is moreover usual to call the co-ordinate OM, the *abscissa*, and MP, the *ordinate*, of the point P.

ON THE CONNEXION OF THE SIGNS OF ALGEBRAICAL AND THE DIRECTIONS OF GEOMETRICAL MAGNITUDES.

It is shown in algebra, that if, through misapprehension of a problem, we measure in one direction, a line which ought to lie in the exactly opposite direction, or if such a mistake be a consequence of some previous misconstruction of the figure, any attempt to deduce the length of that line by algebraical reasoning, will give a negative quantity as the result. And conversely it may be proved by any number of examples, that when an equation in which a occurs has been deduced strictly on the supposition that a is a line measured in one direction, a change of sign in a will turn the equation into that which would have been deduced by the same reasoning, had we begun by measuring the line a in the contrary direction. Hence the change of $+a$ into $-a$, or of $-a$ into $+a$, corresponds in geometry to a change of direction of the line represented by a, and *vice versa*.

In illustration of this general fact, the following problem may be useful. Having a circle of given radius, whose centre is in the intersection of the axes of x and y, and also a straight line cutting the axes in two given points, required the co-ordinates of the points (if any) in which the straight line cuts the circle. Let OA, the radius of the circle $=r$, OE $=a$, OF $=b$, and let the co-ordinates of P, one of the points of intersection required, be OM $=x$, MP $=y$. (Fig. 3.) The point P being in the circle whose radius is r, we have from the right-angled triangle OMP, $x^2+y^2=r^2$, which equation belongs to the co-ordinates of every point in the circle, and is called

the equation of the circle. Again, EM : MP : : EO : OF by similar triangles; or $a-x:y::a:b$, whence $ay+bx=ab$, which is true, by similar reasoning, for every point of the line EF. But for a point P′ lying in EF produced, we have EM′ : M′P′ : : EO : OF, or $x+a:y::a:b$, whence $ay-bx=ab$, an equation which may be obtained from the former by changing the sign of x; and it is evident that the direction of x, in the



second case, is opposite to that in the first. Again, for a point P″ in FE produced, we have EM″ : M″P″ : : EO : OF, or $x-a:y::a:b$, whence $bx-ay=ab$, which may be deduced from the first by changing the sign of y; and it is evident that y is measured in different directions in the first and third cases. Hence the equation $ay+bx=ab$ belongs to all parts of the straight line EF, if we agree to consider M″P″ as negative, when MP is positive, and OM′ as negative

when OM is positive. Thus, if OE $=4$, and OF $=5$, and OM $=1$, we can determine MP from the equation $ay+bx=ab$, or $4y+5=20$, which gives y or MP $=3\frac{3}{4}$. But if OM′ be 1 in length, we can determine M′P′ either by calling OM′, 1, and using the equation $ay-bx=ab$, or calling OM′, -1, and using the equation $ay+bx=ab$, as before. Either gives M′P′ $=6\frac{1}{4}$. The latter method is preferable, inasmuch as it enables us to contain, in one investigation, all the different cases of a problem.

We shall proceed to show that this may be done in the present instance. We have to determine the co-ordinates of the point P, from the following equations:

$$ay+bx=ab,$$
$$x^2+y^2=r^2.$$

Substituting in the second the value of y derived from the first, or $b\left(\frac{a-x}{a}\right)$, we have

$$x^2+b^2\,\frac{(a-x)^2}{a^2}=r^2,$$

$$\text{or } (a^2+b^2)x^2-2ab^2x+a^2(b^2-r^2)=0;$$

and proceeding in a similar manner to find y, we have

$$(a^2+b^2)y^2-2a^2by+b^2(a^2-r^2)=0,$$

which give

$$x=a\,\frac{b^2\pm\sqrt{(a^2+b^2)r^2-a^2b^2}}{a^2+b^2},$$

$$y=b\,\frac{a^2\mp\sqrt{(a^2+b^2)r^2-a^2b^2}}{a^2+b^2};$$

the upper or the lower sign to be taken in both. Hence when $(a^2+b^2)r^2>a^2b^2$, that is, when r is greater than the perpendicular let fall from O upon EF, which perpendicular is

$$\frac{ab}{\sqrt{a^2+b^2}},$$

there are two points of intersection. When $(a^2+b^2)r^2 = a^2b^2$, the two values of x become equal, and also those of y, and there is only one point in which the straight line meets the circle; in this case EF is a tangent to the circle. And if $(a^2+b^2)r^2 < a^2b^2$, the values of x and y are impossible, and the straight line does not meet the circle.

Of these three cases, we confine ourselves to the first, in which there are two points of intersection. The product of the values of x, with their proper sign, is *

$$a^2\frac{b^2-r^2}{a^2+b^2},$$

and of y,

$$b^2\frac{a^2-r^2}{a^2+b^2},$$

the signs of which are the same as those of b^2-r^2, and a^2-r^2. If b and a be both $> r$, the two values of x have the same sign; and it will appear from the figure, that the lines they represent are measured in the same direction. And this whether b and a be positive or negative, since b^2-r^2 and a^2-r^2 are both positive when a and b are numerically greater than r, whatever their signs may be. That is, if our rule, connecting the signs of algebraical and the directions of geometrical magnitudes, be true, let the directions of OE and OF be altered in any way, so long as OE and OF are both greater than OA, the two values of OM will have the same direction, and also those of MP. This result may easily be verified from the figure.

* See *Study of Mathematics* (Chicago: The Open Court Pub. Co.), page 136.

Again, the values of x and y having the same sign, that sign will be (see the equations) the same as that of $2ab^2$ for x, and of $2a^2b$ for y, or the same as that of a for x and of b for y. That is, when OE and OF are both greater than OA, the direction of each set of co-ordinates will be the same as those of OE and OF, which may also be readily verified from the figure.

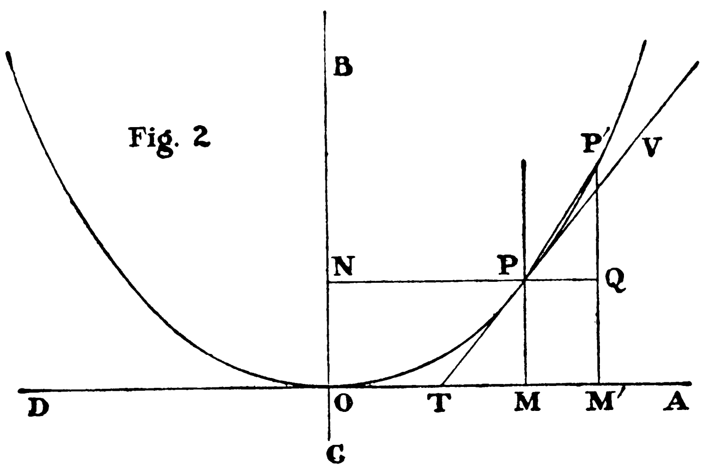
Many other verifications might thus be obtained of the same principle, viz., that any equation which corresponds to, and is true for, all points in the angle AOB, may be used without error for all points lying in the other three angles, by substituting the proper numerical values, with a negative sign, for those co-ordinates whose directions are opposite to those of the co-ordinates in the angle AOB. In this manner, if four points be taken similarly situated in the four angles, the numerical values of whose co-ordinates are $x=4$ and $y=6$, and if the co-ordinates of that point which lies in the angle AOB, are called $+4$ and $+6$; those of the points lying in the angle BOC will be -4 and $+6$; in the angle COD -4 and -6; and in the angle DOE $+4$ and -6.

To return to Fig. 2, if, after having completed the branch of the curve which lies on the right of BC, and whose equation is $y=x^2$, we seek that which lies on the left of BC, we must, by the principles established, substitute $-x$ instead of x, when the numerical value obtained for $(-x)^2$ will be that of y, and the sign will show whether y is to be measured in a similar or contrary direction to that of MP. Since $(-x)^2 = x^2$, the direction and value of y, for a given value of x, remains the same as on the right of BC; whence the remaining branch of the curve is similar and equal in all respects to OP, only lying in the angle BOD.

And thus, if y be any function of x, we can obtain a geometrical representation of the same, by making y the ordinate, and x the abscissa of a curve, every ordinate of which shall be the linear representation of the numerical value of the given function corresponding to the numerical value of the abscissa, the linear unit being a given line.

THE DRAWING OF A TANGENT TO A CURVE.

If the point P (Fig. 2), setting out from O, move along the branch OP, it will continually change the



direction of its motion, never moving, at one point, in the direction which it had at any previous point. Let the moving point have reached P, and let $OM = x$, $MP = y$. Let x receive the increment $MM' = dx$, in consequence of which y or MP becomes M′P′, and receives the increment $QP' = dy$; so that $x + dx$ and $y + dy$ are the co-ordinates of the moving point P, when it arrives at P′. Join PP′, which makes, with PQ or OM, an angle, whose tangent is $\frac{P'Q}{PQ}$ or $\frac{dy}{dx}$. Since the relation $y = x^2$ is true for the co-ordinates of every point in the curve, we have $y + dy = (x + dx)^2$,

the subtraction of the former equation from which gives $dy = 2x\,dx + (dx)^2$, or $\frac{dy}{dx} = 2x + dx$. If the point P′ be now supposed to move backwards towards P, the chord PP′ will diminish without limit, and the inclination of PP′ to PQ will also diminish, but not without limit, since the tangent of the angle P′PQ, or $\frac{dy}{dx}$, is always greater than the limit $2x$. If, therefore, a line PV be drawn through P, making with PQ an angle whose tangent is $2x$, the chord PP′ will, as P′ approaches towards P, or as dx is diminished, continually approximate towards PV, so that the angle P′PV may be made smaller than any given angle, by sufficiently diminishing dx. And the line PV cannot again meet the curve on the side of PP′, nor can any straight line be drawn between it and the curve, the proof of which we leave to the student.

Again, if P′ be placed on the other side of P, so that its co-ordinates are $x - dx$ and $y - dy$, we have $y - dy = (x - dx)^2$, which, subtracted from $y = x^2$, gives $dy = 2x\,dx - (dx)^2$, or $\frac{dy}{dx} = 2x - dx$. By similar reasoning, if the straight line PT be drawn in continuation of PV, making with PN an angle, whose tangent is $2x$, the chord PP′ will continually approach to this line, as before.

The line TPV indicates the direction in which the point P is proceeding, and is called the *tangent* of the curve at the point P. If the curve were the interior of a small solid tube, in which an atom of matter were made to move, being projected into it at O, and if all the tube above P were removed, the line PV is in the direction which the atom would take on emerging at P, and is the line which it would describe. The an-

gle which the tangent makes with the axis of x in any curve, may be found by giving x an increment, finding the ratio which the corresponding increment of y bears to that of x, and determining the limit of that ratio, or the *differential coefficient*. This limit is the trigonometrical tangent* of the angle which the geometrical tangent makes with the axis of x. If $y = \varphi x$, $\varphi' x$ is this trigonometrical tangent. Thus, if the curve be such that the ordinates are the Naperian logarithms† of the abscissæ, or $y = \log x$, and $y + dy = \log x + \frac{1}{x} dx - \frac{1}{2x^2} dx^2$, etc., the geometrical tangent of any point whose abscissa is x, makes with the axis an angle whose trigonometrical tangent is $\frac{1}{x}$.

This problem, of drawing a tangent to any curve, was one, the consideration of which gave rise to the methods of the Differential Calculus.

RATIONAL EXPLANATION OF THE LANGUAGE OF LEIBNITZ.

As the peculiar language of the theory of infinitely small quantities is extensively used, especially in works of natural philosophy, it has appeared right to us to introduce it, in order to show how the terms which are used may be made to refer to some natural and rational mode of explanation. In applying this language to Fig. 2, it would be said that the curve OP is a polygon consisting of an infinite number of

* There is some confusion between these different uses of the word tangent. The geometrical tangent is, as already defined, the line between which and a curve no straight line can be drawn; the trigonometrical tangent has reference to an angle, and is the ratio which, in any right-angled triangle, the side opposite the angle bears to that which is adjacent.

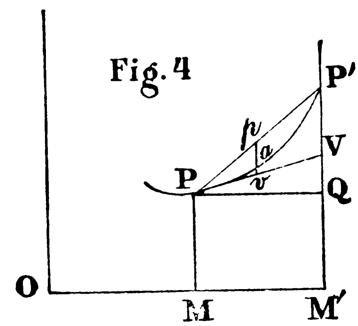
† It may be well to notice that in analysis the Naperian logarithms are the only ones used; while in practice the common, or Briggs's logarithms, are always preferred.

infinitely small sides, each of which produced is a tangent to the curve; also that if MM′ be taken infinitely small, the chord and arc PP′ coincide with one of these rectilinear elements; and that an infinitely small arc coincides with its chord. All which must be interpreted to mean that, the chord and arc being diminished, approach more and more nearly to a ratio of equality as to their lengths; and also that the greatest separation between an arc and its chord may be made as small a part as we please of the whole chord or arc, by sufficiently diminishing the chord.

We shall proceed to a strict proof of this; but in the meanwhile, as a familiar illustration, imagine a small arc to be cut off from a curve, and its extremities joined by a chord, thus forming an arch, of which the chord is the base. From the middle point of the chord, erect a perpendicular to it, meeting the arc, which will thus represent the height of the arch. Imagine this figure to be magnified, without distortion or alteration of its proportions, so that the larger figure may be, as it is expressed, a true picture of the smaller one. However the original arc may be diminished, let the magnified base continue of a given length. This is possible, since on any line a figure may be constructed similar to a given figure. If the original curve could be such that the height of the arch could never be reduced below a certain part of the chord, say one thousandth, the height of the magnified arch could never be reduced below one thousandth of the magnified chord, since the proportions of the two figures are the same. But if, in the original curve, an arc can be taken so small that the height of the arch is as small a part as we please of the chord, it will follow that in the magnified figure where

the chord is always of one length, the height of the arch can be made as small as we please, seeing that it can be made as small a part as we please of a given line. It is possible in this way to conceive a whole curve so magnified, that a given arc, however small, shall be represented by an arc of any given length, however great; and the proposition amounts to this, that let the dimensions of the magnified curve be any given number of times the original, however great, an arch can be taken upon the original curve so small, that the height of the corresponding arch in the magnified figure shall be as small as we please.

Fig. 4



Let PP' (Fig. 4) be a part of a curve, whose equation is $y = \varphi(x)$, that is, PM may always be found by substituting the numerical value of OM in a given function of x. Let $OM = x$ receive the increment $MM' = dx$, which we may afterwards suppose as small as we please, but which, in order to render the figure more distinct, is here considerable. The value of PM or y is φx, and that of P'M' or $y + dy$ is $\varphi(x + dx)$.

Draw PV, the tangent at P, which, as has been shown, makes, with PQ, an angle, whose trigonometrical tangent is the limit of the ratio $\frac{dy}{dx}$, when x is decreased, or $\varphi' x$. Draw the chord PP', and from any

point in it, for example, its middle point p, draw pv parallel to PM, cutting the curve in a. The value of

$$\text{P}'\text{Q}, \text{ or } dy, \text{ or } \varphi(x+dx)-\varphi x \text{ is}$$

$$\text{P}'\text{Q}=\varphi' x\, dx+\varphi'' x\,\frac{(dx)^2}{2}+\varphi''' x\,\frac{(dx)^3}{2.3}+\text{etc.}$$

But $\varphi' x\, dx$ is $\tan \text{VPQ}.\text{PQ}=\text{VQ}$. Hence VQ is the first term of this series, and P′V the aggregate of the rest. But it has been shown that dx can be taken so small, that any one term of the above series shall contain the rest, as often as we please. Hence PQ can be taken so small that VQ shall contain VP′ as often as we please, or the ratio of VQ to VP′ shall be as great as we please. And the ratio VQ to PQ continues finite, being always $\varphi' x$; hence P′V also decreases without limit as compared with PQ.

Next, the chord PP′ or $\sqrt{(dx)^2+(dy)^2}$, or

$$dx\sqrt{1+\left(\frac{dy}{dx}\right)^2},$$

is to PQ or dx in the ratio of $\sqrt{1+\left(\frac{dy}{dx}\right)^2}:1$, which, as PQ is diminished, continually approximates to that of $\sqrt{1+(\varphi' x)^2}:1$, which is the ratio of PV:PQ. Hence the ratio of PP′:PV continually approaches to unity, or PQ may be taken so small that the difference of PP′ and PV shall be as small a part of either of them as we please.

Finally, the arc PP′ is greater than the chord PP′ and less than PV + VP′. Hence $\frac{\text{arc PP}'}{\text{chord PP}'}$ lies between 1 and $\frac{\text{PV}}{\text{PP}'}+\frac{\text{VP}'}{\text{PP}'}$, the former of which two fractions can be brought as near as we please to unity, and the latter can be made as small as we please; for

since P′V can be made as small a part of PQ as we please, still more can it be made as small a part as we please of PP′, which is greater than PQ. Therefore the arc and chord PP′ may be made to have a ratio as nearly equal to unity as we please. And because pa is less than pv, and therefore less than P′V, it follows that pa may be made as small a part as we please of PQ, and still more of PP′.

In these propositions is contained the rational explanation of the proposition of Leibnitz, that "an infinitely small arc is equal to, and coincides with, its chord."

ORDERS OF INFINITY.

Let there be any number of series, arranged in powers of h, so that the lowest power is first; let them contain none but whole powers, and let them all be such, that each will be convergent, on giving to h a sufficiently small value: as follows,

$$\begin{array}{rrrrrl r}
Ah + & Bh^2 + & Ch^3 + & Dh^4 + & Eh^5 + & \text{etc.} & (1)\\
 & B'h^2 + & C'h^3 + & D'h^4 + & E'h^5 + & \text{etc.} & (2)\\
 & & C''h^3 + & D''h^4 + & E''h^5 + & \text{etc.} & (3)\\
 & & & D'''h^4 + & E'''h^5 + & \text{etc.} & (4)\\
 & & & & & \text{etc.} & \text{etc.}
\end{array}$$

As h is diminished, all these expressions decrease without limit; but the first *increases* with respect to the second, that is, contains it more times after a decrease of h, than it did before. For the ratio of (1) to (2) is that of $A + Bh + Ch^2 +$ etc. to $B'h + C'h^2$ $+$ etc., the ratio of the two not being changed by dividing both by h. The first term of the latter ratio approximates continually to A, as h is diminished, and the second can be made as small as we please, and therefore can be contained in the first as often as

we please. Hence the ratio (1) to (2) can be made as great as we please. By similar reasoning, the ratio (2) to (3), of (3) to (4), etc., can be made as great as we please. We have, then, a series of quantities, each of which, by making h sufficiently small, can be made as small as we please. Nevertheless this decrease increases the ratio of the first to the second, of the second to the third, and so on, and the increase is without limit.

Again, if we take (1) and h, the ratio of (1) to h is that of $A + Bh + Ch^2 +$ etc. to 1, which, by a sufficient decrease of h, may be brought as near as we please to that of A to 1. But if we take (1) and h^2, the ratio of (1) to h^2 is that of $A + Bh +$ etc. to h, which, by previous reasoning, may be increased without limit; and the same for any higher power of h. Hence (1) is said to be *comparable* to the first power of h, or *of the first order*, since this is the only power of h whose ratio to (1) tends towards a finite limit. By the same reasoning, the ratio of (2) to h^2, which is that of $B' + C'h +$ etc. to 1, continually approaches that of B' to 1; but the ratio (2) to h, which is that of $B'h + C'h^2 +$ etc. to 1, diminishes without limit, as h is decreased, while the ratio of (2) to h^3, or of $B' + C'h +$ etc. to h, increases without limit. Hence (2) is said to be *comparable* to the second power of h, or *of the second order*, since this is the only power of h whose ratio to (2) tends towards a finite limit. In the language of Leibnitz, if h be an infinitely small quantity, (1) is an infinitely small quantity of the first order, (2) is an infinitely small quantity of the second order, and so on.

We may also add that the ratio of two series of the same order continually approximates to the ratio

of their lowest terms. For example, the ratio of $Ah^3 + Bh^4 +$ etc. to $A'h^3 + B'h^4 +$ etc. is that of $A + Bh +$ etc. to $A' + B'h +$ etc., which, as h is diminished, continually approximates to the ratio of A to A′, which is also that of Ah^3 to $A'h^3$, or the ratio of the lowest terms. In Fig. 4, PQ or dx being put in place of h, QP′, or $\varphi' x \, dx + \varphi'' x \dfrac{(dx)^2}{2}$, etc., is of the first order, as are PV, and the chord PP′; while P′V, or $\varphi'' x \dfrac{(dx)^2}{2} +$ etc., is of the second order.

The converse proposition is readily shown, that if the ratio of two series arranged in powers of h continually approaches to some finite limit as h is diminished, the two series are of the same order, or the exponent of the lowest power of h is the same in both. Let Ah^a and Bh^b be the lowest powers of h, whose ratio, as has just been shown, continually approximates to the actual ratio of the two series, as h is diminished. The hypothesis is that the ratio of the two series, and therefore that of Ah^a to Bh^b, has a finite limit. This cannot be if $a > b$, for then the ratio of Ah^a to Bh^b is that of Ah^{a-b} to B, which diminishes without limit; neither can it be when $a < b$, for then the same ratio is that of A to Bh^{b-a}, which increases without limit; hence a must be equal to b.

We leave it to the student to prove strictly a proposition assumed in the preceding; viz., that if the ratio of P to Q has unity for its limit, when h is diminished, the limiting ratio of P to R will be the same as the limiting ratio of Q to R. We proceed further to illustrate the Differential Calculus as applied to Geometry.

A GEOMETRICAL ILLUSTRATION.

Let OC and OD (Fig. 5) be two axes at right angles to one another, and let a line AB of given length be placed with one extremity in each axis. Let this line move from its first position into that of A'B' on one side, and afterwards into that of A''B'' on the other side, always preserving its first length. The motion of a ladder, one end of which is against a wall, and the other on the ground, is an instance.

Let A'B' and A''B'' intersect AB in P' and P''. If A''B'' were gradually moved from its present position into that of A'B', the point P'' would also move grad-

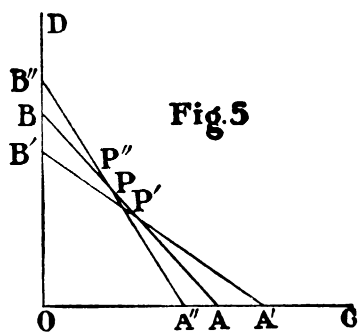


Fig. 5

ually from its present position into that of P', passing, in its course, through every point in the line P'P''. But here it is necessary to remark that AB is itself one of the positions intermediate between A' B' and A'' B'', and when two lines are, by the motion of one of them, brought into one and the same straight line, they intersect one another (if this phrase can be here applied at all) in every point, and all idea of one distinct point of intersection is lost. Nevertheless P'' describes one part of P''P' before A'' B'' has come into the position AB, and the rest afterwards, when it is between AB and A' B'.

Let P be the point of separation ; then every point of P′P″, except P, is a real point of intersection of AB, with one of the positions of A″B″, and when A″B″ has moved very near to AB, the point P″ will be very near to P ; and there is no point so near to P, that it may not be made the intersection of A″B″ and AB, by bringing the former sufficiently near to the latter. This point P is, therefore, the *limit* of the intersections of A″B″ and AB, and cannot be found by the ordinary application of algebra to geometry, but may be made the subject of an inquiry similar to those

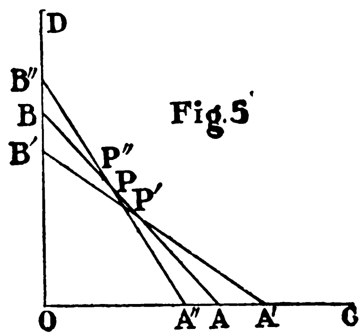


Fig. 5

which have hitherto occupied us, in the following manner:

Let OA $=a$, OB $=b$, AB $=$ A′B′ $=$ A″B″ $=l$. Let AA′ $=da$, BB′ $=db$, whence OA′ $=a+da$, OB′ $=b-db$. We have then $a^2+b^2=l^2$, and $(a+da)^2+(b-db)^2=l^2$; subtracting the former of which from the development of the latter, we have

$$2a\,da+(da)^2-2b\,db+(db)^2=0$$

$$\text{or } \frac{db}{da}=\frac{2a+da}{2b-db} \qquad (1)$$

As A′B′ moves towards AB, da and db are diminished without limit, a and b remaining the same; hence the limit of the ratio $\frac{db}{da}$ is $\frac{2a}{2b}$ or $\frac{a}{b}$.

Let the co-ordinates* of P′ be OM′ $=x$ and M′P, $=y$. Then (page 32) the co-ordinates of any point in AB have the equation

$$ay+bx=ab \tag{2}$$

The point P′ is in this line, and also in the one which cuts off $a+da$ and $b-db$ from the axes, whence

$$(a+da)y+(b-db)x=(a+da)(b-db) \tag{3}$$

subtract (2) from (3) after developing the latter, which gives

$$y\,da-x\,db=b\,da-a\,db-da\,db \tag{4}$$

If we now suppose A′B′ to move towards AB, equation (4) gives no result, since each of its terms diminishes without limit. If, however, we divide (4) by da, and substitute in the result the value of $\frac{db}{da}$ obtained from (1) we have

$$y-x\,\frac{2a+da}{2b-db}=b-a\,\frac{2a+da}{2b-db}-db \tag{5}$$

From this and (2) we might deduce the values of y and x, for the point P′, as the figure actually stands. Then by diminishing db and da without limit, and observing the limit towards which x and y tend, we might deduce the co-ordinates of P, the limit of the intersections.

The same result may be more simply obtained, by diminishing da and db in equation (5), before obtaining the values of y and x. This gives

$$y-\frac{a}{b}\,x=b-\frac{a^2}{b} \text{ or } by-ax=b^2-a^2 \tag{6}$$

From (6) and (2) we find (Fig. 6)

$$x=\text{OM}=\frac{a^3}{a^2+b^2}=\frac{a^3}{l^2} \text{ and } y=\text{MP}=\frac{b^3}{a^2+b^2}=\frac{b^3}{l^2}.$$

* The lines OM′ and M′P′ are omitted, to avoid crowding the figure.

This limit of the intersections is different for every different position of the line AB, but may be determined, in every case, by the following simple construction.

Since (Fig. 6) BP : PN, or OM : : BA : AO, we have $BP = OM \frac{BA}{AO} = \frac{a^3}{l^2}\frac{l}{a} = \frac{a^2}{l}$; and, similarly, $PA = \frac{b^2}{l}$. Let OQ be drawn perpendicular to BA; then since OA is a mean proportional between AQ and AB, we have $AQ = \frac{a^2}{l}$, and similarly $BQ = \frac{b^2}{l}$. Hence BP = AQ and AP = BQ, or the point P is as far from either extremity of AB as Q is from the other.

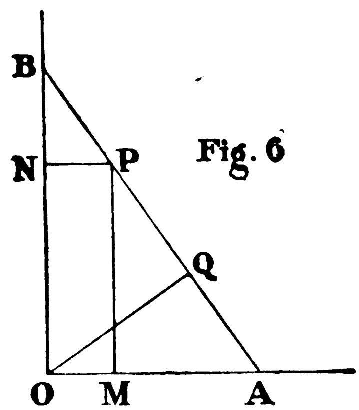


Fig. 6

THE SAME PROBLEM SOLVED BY THE PRINCIPLES OF LEIBNITZ.

We proceed to solve the same problem, using the principles of Leibnitz, that is, supposing magnitudes can be taken so small, that those proportions may be regarded as absolutely correct, which are not so in reality, but which only approach more nearly to the truth, the smaller the magnitudes are taken. The inaccuracy of this supposition has been already pointed out; yet it must be confessed that this once got over,

the results are deduced with a degree of simplicity and consequent clearness, not to be found in any other method. The following cannot be regarded as a demonstration, except by a mind so accustomed to the subject that it can readily convert the various inaccuracies into their corresponding truths, and see, at one glance, how far any proposition will affect the final result. The beginner will be struck with the extraordinary assertions which follow, given in their most naked form, without any attempt at a less startling mode of expression.

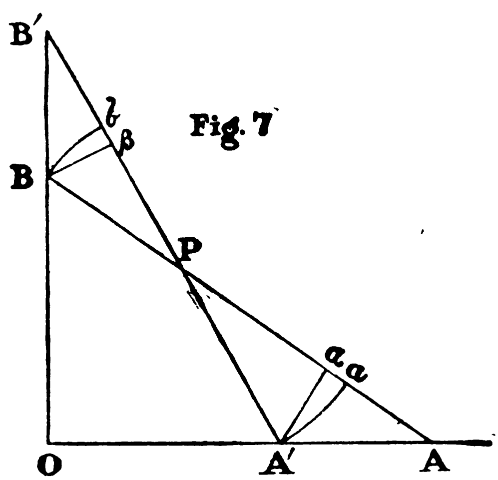


Fig. 7

Let A′B′ (Fig. 7) be a position of AB infinitely near to it; that is, let A′PA be an infinitely small angle. With the centre P, and the radii PA′ and PB, describe the infinitely small arcs A′*a*, B*b*. An infinitely small arc of a circle is a straight line perpendicular to its radius; hence A′*a*A and B*b*B′ are right-angled triangles, the first similar to BOA, the two having the angle A in common, and the second similar to B′OA′. Again, since the angles of BOA, which are finite, only differ from those of B′OA′ by the infinitely small angle A′PA, they may be regarded as

equal; whence $A'aA$ and $B'bB$ are similar to BOA, and to one another. Also P is the point of which we are in search, or infinitely near to it; and since $BA = B'A'$, of which $BP = bP$ and $aP = A'P$, the remainders $B'b$ and Aa are equal. Moreover, Bb and $A'a$ being arcs of circles subtending equal angles, are in the proportion of the radii BP and PA'.

Hence we have the following proportions:

$$Aa : A'a :: OA : OB :: a : b$$
$$Bb : B'b :: OA : OB :: a : b.$$

The composition of which gives, since $Aa = B'b$:

$$Bb : A'a :: a^2 : b^2.$$

Also $$Bb : A'a :: BP : Pa,$$

whence $$BP : Pa :: a^2 : b^2,$$

and $$BP + Pa : Pa :: a^2 + b^2 : b^2.$$

But Pa only differs from PA by the infinitely small quantity Aa, and $BP + PA = l$, and $a^2 + b^2 = l^2$; whence

$$l : PA :: l^2 : b^2, \quad \text{or } PA = \frac{b^2}{l},$$

which is the result already obtained.

In this reasoning we observe four independent errors, from which others follow: (1) that Bb and $A'a$ are straight lines at right-angles to Pa; (2) that BOA B'OA' are similar triangles; (3) that P is really the point of which we are in search; (4) that PA and Pa are equal. But at the same time we observe that every one of these assumptions approaches the truth, as we diminish the angle A'PA, so that there is no magnitude, line or angle, so small that the linear or angular errors, arising from the above-mentioned suppositions, may not be made smaller.

We now proceed to put the same demonstration

in a stricter form, so as to neglect no quantity during the process. This should always be done by the beginner, until he is so far master of the subject as to be able to annex to the inaccurate terms the ideas necessary for their rational explanation. To the former figure add $B\beta$ and $A\alpha$, the real perpendiculars, with which the arcs have been confounded. Let $\angle A'PA = d\theta$, $PA = p$, $Aa = dp$, $BP = q$, $B'b = dq$; and $OA = a$, $OB = b$, and $AB = l$. Then* $A'a = (p - dp)\,d\theta$, $Bb = q\,d\theta$, and the triangles $A'A\alpha$ and $B'B\beta$ are similar to

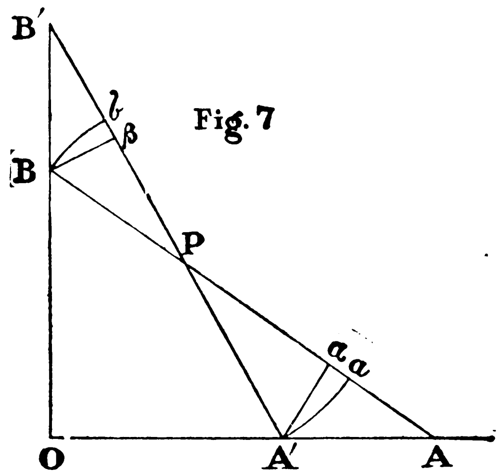


Fig. 7

BOA and B′OA′. The perpendiculars $A'\alpha$ and $B\beta$ are equal to $PA' \sin d\theta$ and $PB \sin d\theta$, or $(p - dp)\sin d\theta$ and $q \sin d\theta$. Let $a\alpha = \mu$ and $b\beta = \nu$. These (p. 9) will diminish without limit as compared with $A'\alpha$ and $B\beta$; and since the ratios of $A'\alpha$ to αA and $B\beta$ to $\beta B'$ continue finite (these being sides of triangles similar to AOB and A′OB′), $a\alpha$ and $b\beta$ will diminish indefinitely with respect to αA and $\beta B'$. Hence the ratio $A\alpha$ to $\beta B'$ or $dp + \mu$ to $dq + \nu$ will continually approximate to that of dp to dq, or a ratio of equality.

* For the unit employed in measuring an angle, see *Study of Mathematics* (Chicago, 1898), pages 273–277.

The exact proportions, to which those in the last page are approximations, are as follows:

$$
\begin{array}{llll}
dp+\mu : (p-dp)\sin d\theta & & :: a & : b, \\
q\sin d\theta : & dq+\nu & :: a-da & : b+db;
\end{array}
$$

by composition of which, recollecting that $dp=dq$ (which is rigorously true) and dividing the two first terms of the resulting proportion by dp, we have

$$q\left(1+\frac{\mu}{dp}\right) : (p-dp)\left(1+\frac{\nu}{dp}\right) :: a(a-da) : b(b+db).$$

If $d\theta$ be diminished without limit, the quantities da, db, and dp, and also the ratios $\frac{\mu}{dp}$ and $\frac{\nu}{dp}$, as above-mentioned, are diminished without limit, so that the limit of the proportion just obtained, or the proportion which gives the limits of the lines into which P divides AB, is

$$q : p :: a^2 : b^2,$$

hence $\quad q+p=l : p :: a^2+b^2=l^2 : b^2,$

the same as before.

AN ILLUSTRATION FROM DYNAMICS.

We proceed to apply the preceding principles to dynamics, or the theory of motion.

Suppose a point moving along a straight line uniformly; that is, if the whole length described be divided into any number of equal parts, however great, each of those parts is described in the same time. Thus, whatever length is described in the first second of time, or in any part of the first second, the same is described in any other second, or in the same part of any other second. The number of units of length described in a unit of time is called the *velocity*; thus a velocity of 3·01 feet in a second means that the

point describes three feet and one hundredth in each second, and a proportional part of the same in any part of a second. Hence, if v be the velocity, and t the units of time elapsed from the beginning of the motion, vt is the length described; and if any length described be known, the velocity can be determined by dividing that length by the time of describing it. Thus, a point which moves uniformly through 3 feet in $1\frac{1}{2}$ second, moves with a velocity of $3 \div 1\frac{1}{2}$, or 2 feet per second.

Let the point not move uniformly; that is, let different parts of the line, having the same length, be described in different times; at the same time let the motion be *continuous*, that is, not suddenly increased or decreased, as it would be if the point were composed of some hard matter, and received a blow while it was moving. This will be the case if its motion be represented by some algebraical function of the time, or if, t being the number of units of time during which the point has moved, the number of units of length described can be represented by φt. This, for example, we will suppose to be $t + t^2$, the unit of time being one second, and the unit of length one inch; so that $\frac{1}{2} + \frac{1}{4}$, or $\frac{3}{4}$ of an inch, is described in the first half second; $1 + 1$, or two inches, in the first second; $2 + 4$, or six inches, in the first two seconds, and so on.

Here we have no longer an evident measure of the velocity of the point; we can only say that it obviously increases from the beginning of the motion to the end, and is different at every two different points. Let the time t elapse, during which the point will describe the distance $t + t^2$; let a further time dt elapse, during which the point will increase its distance to $t + dt + (t + dt)^2$, which, diminished by $t + t^2$, gives

$dt + 2t\,dt + (dt)^2$ for the length described during the increment of time dt. This varies with the value of t; thus, in the interval dt after the first second, the length described is $3dt + dt^2$; after the second second, it is $5dt + (dt)^2$, and so on. Nor can we, as in the case of uniform motion, divide the length described by the time, and call the result the velocity with which that length is described; for no length, however small, is here uniformly described. If we were to divide a length by the time in which it is described, and also its first and second halves by the times in which they are respectively described, the three results would be all different from one another.

Here a difficulty arises, similar to that already noticed, when a point moves along a curve; in which, as we have seen, it is improper to say that it is moving in any one direction through an arc, however small. Nevertheless a straight line was found at every point, which did, more nearly than any other straight line, represent the direction of the motion. So, in this case, though it is incorrect to say that there is any uniform velocity with which the point continues to move for any portion of time, however small, we can, at the end of every time, assign a uniform velocity, which shall represent, more nearly than any other, the rate at which the point is moving. If we say that, at the end of the time t, the point is moving with a velocity v, we must not now say that the length vdt is described in the succeeding interval of time dt; but we mean that dt may be taken so small, that vdt shall bear to the distance actually described a ratio as near to equality as we please.

Let the point have moved during the time t, after which let successive intervals of time elapse, each

equal to dt. At the end of the times, t, $t+dt$, $t+2dt$, $t+3dt$, etc., the whole lengths described will be $t+t^2$, $t+dt+(t+dt)^2$, $t+2dt+(t+2dt)^2$, $t+3dt+(t+3dt)^2$, etc.; the differences of which, or $dt+2t\,dt+(dt)^2$, $dt+2t\,dt+3(dt)^2$, $dt+2t\,dt+5(dt)^2$, etc., are the lengths described in the first, second, third, etc., intervals dt. These are not equal to one another, as would be the case if the velocity were uniform; but by making dt sufficiently small, their ratio may be brought as near to equality as we please, since the terms $(dt)^2$, $3(dt)^2$, etc., by which they all differ from the common part $(1+2t)\,dt$, may be made as small as we please, in comparison of this common part. If we divide the above-mentioned lengths by dt, which does not alter their ratio, they become $1+2t+dt$, $1+2t+3dt$, $1+2t+5dt$, etc., which may be brought as near as we please to equality, by sufficient diminution of dt. Hence $1+2t$ is said to be the velocity of the point after the time t; and if we take a succession of equal intervals of time, each equal to dt, and sufficiently small, the lengths described in those intervals will bear to $(1+2t)\,dt$, the length which would be described in the same interval with the uniform velocity $1+2t$, a ratio as near to equality as we please. And observe, that if φt is $t+t^2$, $\varphi' t$ is $1+2t$, or the coefficient of h in $(t+h)+(t+h)^2$.

In the same way it may be shown, that if the point moves so that φt always represents the length described in the time t, the differential coefficient of φt or $\varphi' t$, is the velocity with which the point is moving at the end of the time t. For the time t having elapsed, the whole lengths described at the end of the times t and $t+dt$ are φt and $\varphi(t+dt)$; whence the length described during the time dt is

$$\varphi(t+dt)-\varphi t, \text{ or } \varphi' t\, dt+\varphi'' t\frac{(dt)^2}{2}+\text{etc.}$$

Similarly, the length described in the next interval dt is

$$\varphi(t+2dt)-\varphi(t+dt)\text{; or,}$$

$$\varphi t+\varphi' t\, 2dt+\varphi'' t\frac{(2dt)^2}{2}+\text{etc.}$$

$$-\left(\varphi t+\varphi' t\, dt+\varphi'' t\frac{(dt)^2}{2}+\text{etc.}\right),$$

which is

$$\varphi' t\, dt+3\varphi'' t\frac{(dt)^2}{2}+\text{etc.;}$$

the length described in the third interval dt is $\varphi' t\, dt + 5\varphi'' t\frac{(dt)^2}{2}+\text{etc., etc.}$

Now, it has been shown for each of these, that the first term can be made to contain the aggregate of all the rest as often as we please, by making dt sufficiently small; this first term is $\varphi' t\, dt$ in all, or the length which would be described in the time dt by the velocity $\varphi' t$ continued uniformly: it is possible, therefore, to take dt so small, that the lengths actually described in a succession of intervals equal to dt, shall be as nearly as we please in a ratio of equality with those described in the same intervals of time by the velocity $\varphi' t$. For example, it is observed in bodies which fall to the earth from a height above it, when the resistance of the air is removed, that if the time be taken in seconds, and the distance in feet, the number of feet fallen through in t seconds is always at^2, where $a = 16\frac{1}{12}$ very nearly; what is the velocity of a body which has fallen *in vacuo* for four seconds? Here φt being at^2, we find, by substituting $t+h$, or $t+dt$, instead of t, that $\varphi' t$ is $2at$, or $2\times 16\frac{1}{12}\times t$; which, at

the end of four seconds, is $32\frac{1}{6} \times 4$, or $128\frac{2}{3}$ feet. That is, at the end of four seconds a falling body moves at the rate of $128\frac{2}{3}$ feet per second. By which we do not mean that it continues to move with this velocity for any appreciable time, since the rate is always varying; but that the length described in the interval dt after the fourth second, may be made as nearly as we please in a ratio of equality with $128\frac{2}{3} \times dt$, by taking dt sufficiently small. This velocity $2at$ is said to be *uniformly* accelerated; since in each second the same velocity $2a$ is gained. And since, when x is the space described, $\varphi' t$ is the limit of $\frac{dx}{dt}$, the velocity is also this limit; that is, when a point does not move uniformly, the velocity is not represented by any increment of length divided by its increment of time, but by the limit to which that ratio continually tends, as the increment of time is diminished.

SIMPLE HARMONIC MOTION.

We now propose the following problem: A point moves uniformly round a circle; with what velocities do the abscissa and ordinate increase or decrease, at any given point? (Fig. 8.)

Let the point P, setting out from A, describe the arc AP, etc., with the uniform velocity of a inches per second. Let $OA = r$, $\angle AOP = \theta$, $\angle POP' = d\theta$, $OM = x$, $MP = y$, $MM' = dx$, $QP' = dy$.

From the first principles of trigonometry

$$x = r\cos\theta$$

$$x - dx = r\cos(\theta + d\theta) = r\cos\theta\cos d\theta - r\sin\theta\sin d\theta$$

$$y = r\sin\theta$$

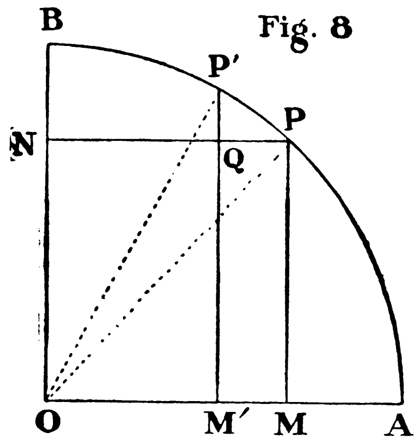
$$y + dy = r\sin(\theta + d\theta) = r\sin\theta\cos d\theta + r\cos\theta\sin d\theta.$$

Subtracting the second from the first, and the third from the fourth, we have

$$dx = r\sin\theta\sin d\theta + r\cos\theta(1-\cos d\theta) \qquad (1)$$
$$dy = r\cos\theta\sin d\theta + r\sin\theta(1-\cos d\theta) \qquad (2)$$

But if $d\theta$ be taken sufficiently small, $\sin d\theta$, and $d\theta$, may be made as nearly in a ratio of equality as we please, and $1-\cos d\theta$ may be made as small a part as we please, either of $d\theta$ or $\sin d\theta$. These follow from Fig. 1, in which it was shown that BM and the arc BA, or (if $OA = r$ and $AOB = d\theta$), $r \sin d\theta$ and $r\,d\theta$, may be brought as near to a ratio of equality as we

Fig. 8



please, which is therefore true of $\sin d\theta$ and $d\theta$. Again, it was shown that AM, or $r - r\cos d\theta$, can be made as small a part as we please, either of BM or the arc BA, that is, either of $r \sin d\theta$, or $r\,d\theta$; the same is therefore true of $1-\cos d\theta$, and either $\sin d\theta$ or $d\theta$. Hence, if we write equations (1) and (2) thus,

$$dx = r\sin\theta\, d\theta \quad (1) \qquad\qquad dy = r\cos\theta\, d\theta \quad (2),$$

we have equations, which, though never exactly true, are such that by making $d\theta$ sufficiently small, the errors may be made as small parts of $d\theta$ as we please. Again, since the arc AP is uniformly described, so also is the angle POA; and since an arc a is described

in one second, the angle $\frac{a}{r}$ is described in the same time; this is, therefore, the *angular velocity.** If we divide equations (1) and (2) by dt, we have

$$\frac{dx}{dt} = r \sin\theta \frac{d\theta}{dt} \qquad \frac{dy}{dt} = r \cos\theta \frac{d\theta}{dt};$$

these become more nearly true as dt and $d\theta$ are diminished, so that if for $\frac{dx}{dt}$, etc., the limits of these ratios be substituted, the equations will become rigorously true. But these limits are the velocities of x, y, and θ, the last of which is also $\frac{a}{r}$; hence

$$\text{velocity of } x = r \sin\theta \times \frac{a}{r} = a \sin\theta,$$

$$\text{velocity of } y = r \cos\theta \times \frac{a}{r} = a \cos\theta;$$

that is, the point M moves towards O with a variable velocity, which is always such a part of the velocity of P, as $\sin\theta$ is of unity, or as PM is of OB; and the distance PM increases, or the point N moves from O, with a velocity which is such a part of the velocity of P as $\cos\theta$ is of unity, or as OM is of OA. [The motion of the point M or the point N is called in physics *a simple harmonic motion.*]

In the language of Leibnitz, the results of the two foregoing sections would be expressed thus: If a point move, but not uniformly, it may still be considered as moving uniformly for any infinitely small

*The same considerations of velocity which have been applied to the motion of a point along a line may also be applied to the motion of a line round a point. If the angle so described be always increased by equal angles in equal portions of time, the angular velocity is said to be uniform, and is measured by the number of angular units described in a unit of time. By similar reasoning to that already described, if the velocity with which the angle increases be not uniform, so that at the end of the time t the angle described is $\theta = \phi t$, the angular velocity is $\phi' t$, or the limit of the ratio $\frac{d\theta}{dt}$.

time; and the velocity with which it moves is the infinitely small space thus described, divided by the infinitely small time.

THE METHOD OF FLUXIONS.

The foregoing process contains the method employed by Newton, known by the name of the *Method of Fluxions*. If we suppose y to be any function of x, and that x increases with a given velocity, y will also increase or decrease with a velocity depending: (1) upon the velocity of x; (2) upon the function which y is of x. These velocities Newton called the fluxions of y and x, and denoted them by $\dot{y}$ and $\dot{x}$. Thus, if $y=x^2$, and if in the interval of time dt, x becomes $x+dx$, and y becomes $y+dy$, we have $y+dy=(x+dx)^2$, and $dy=2x\,dx+(dx)^2$, or $\frac{dy}{dt}=2x\frac{dx}{dt}+\frac{dx}{dt}dx$. If we diminish dt, the term $\frac{dx}{dt}dx$ will diminish without limit, since one factor continually approaches to a given quantity, viz., the velocity of x, and the other diminishes without limit. Hence we obtain the velocity of $y=2x\times$ the velocity of x, or $\dot{y}=2x\,\dot{x}$, which is used in the method of fluxions instead of $dy=2x\,dx$ considered in the manner already described. The processes are the same in both methods, since the ratio of the velocities is the limiting ratio of the corresponding increments, or, according to Leibnitz, the ratio of the infinitely small increments. We shall hereafter notice the common objection to the Method of Fluxions.

ACCELERATED MOTION.

When the velocity of a material point is suddenly increased, an *impulse* is said to be given to it, and the

magnitude of the impulse or impulsive force is in proportion to the velocity created by it. Thus, an impulse which changes the velocity from 50 to 70 feet per second, is twice as great as one which changes it from 50 to 60 feet. When the velocity of the point is altered, not suddenly but continuously, so that before the velocity can change from 50 to 70 feet, it goes through all possible intermediate velocities, the point is said to be acted on by an *accelerating force*. *Force* is a name given to that which causes a change in the velocity of a body. It is said to act uniformly, when the velocity acquired by the point in any one interval of time is the same as that acquired in any other interval of equal duration. It is plain that we cannot, by supposing any succession of impulses, however small, and however quickly repeated, arrive at a uniformly accelerated motion; because the length described between any two impulses will be uniformly described, which is inconsistent with the idea of continually accelerated velocity. Nevertheless, by diminishing the magnitude of the impulses, and increasing their number, we may come as near as we please to such a continued motion, in the same way as, by diminishing the magnitudes of the sides of a polygon, and increasing their number, we may approximate as near as we please to a continous curve.

Let a point, setting out from a state of rest, increase its velocity uniformly, so that in the time t, it may acquire the velocity v—what length will have been described during that time t? Let the time t and the velocity v be both divided into n equal parts, each of which is t' and v'; so that $nt' = t$, and $nv' = v$. Let the velocity v' be communicated to the point at rest; after an interval of t' let another velocity v' be

communicated, so that during the second interval t' the point has a velocity $2v'$; during the third interval let the point have the velocity $3v'$, and so on; so that in the last or n^{th} interval the point has the velocity nv'. The space described in the first interval is, therefore, $v't'$; in the second, $2v't'$; in the third $3v't'$; and so on, till in the n^{th} interval it is $nv't'$. The whole space described is, therefore,

$$v't' + 2v't' + 3v't' + \ldots + (n-1)v't' + nv't'$$

$$\text{or } [1+2+3\ldots\ldots(n-1)+n]\,v't' = n.\frac{n+1}{2}\,v't'$$

$$= \frac{n^2v't' + nv't'}{2}.$$

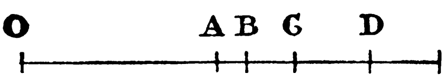
In this substitute v for nv', and t for nt', which gives for the space described $\frac{1}{2}v(t+t')$. The smaller we suppose t', the more nearly will this approach to $\frac{1}{2}vt$. But the smaller we suppose t', the greater must be n, the number of parts into which t is divided; and the more nearly do we render the motion of the point uniformly accelerated. Hence the limit to which we approximate by diminishing t' without limit, is the length described in the time t by a uniformly accelerated velocity, which shall increase from 0 to v in that time. This is $\frac{1}{2}vt$, or half the length which would have been described by the velocity v continued uniformly from the beginning of the motion.

It is usual to measure the accelerating force by the velocity acquired in one second. Let this be g; then since the same velocity is acquired in every other second, the velocity acquired in t seconds will be gt, or $v=gt$. Hence the space described is $\frac{1}{2}gt \times t$, or $\frac{1}{2}gt^2$. If the point, instead of being at rest at the beginning of the acceleration, had had the velocity a, the lengths

described in the successive intervals would have been $at' + v't'$, $at' + 2v't'$, etc.; so that to the space described by the accelerated motion would have been added nat', or at, and the whole length would have been $at + \frac{1}{2}gt^2$. By similar reasoning, had the force been a uniformly *retarding* force, that is, one which diminished the initial velocity a equally in equal times, the length described in the time t would have been $at - \frac{1}{2}gt^2$.

Now let the point move in such a way, that the velocity is accelerated or retarded, but not uniformly; that is, in different times of equal duration, let different velocities be lost or gained. For example, let the point, setting out from a state of rest, move in such a

Fig. 9



way that the number of inches passed over in t seconds is always t^3. Here $\varphi t = t^3$, and the velocity acquired by the body at the end of the time t, is the coefficient of dt in $(t + dt)^3$, or $3t^2$ inches per second. Let the point (Fig. 9) be at A at the end of the time t; and let AB, BC, CD, etc., be lengths described in successive equal intervals of time, each of which is dt. Then the velocities at A, B, C, etc., are $3t^2$, $3(t + dt)^2$, $3(t + 2dt)^2$, etc., and the lengths AB, BC, CD, etc., are $(t + dt)^3 - t^3$, $(t + 2dt)^3 - (t + dt)^3$, $(t + 3dt)^3 - (t + 2dt)^3$, etc.

VELOCITY AT	
A	$3t^2$
B	$3t^2 + 6t\,dt + 3(dt)^2$
C	$3t^2 + 12t\,dt + 12(dt)^2$

LENGTH OF	
AB	$3t^2dt + 3t(dt)^2 + (dt)^3$
BC	$3t^2dt + 9t(dt)^2 + 7(dt)^3$
CD	$3t^2dt + 15t(dt)^2 + 19(dt)^3$

If we could, without error, reject the terms containing $(dt)^2$ in the velocities, and those containing $(dt)^3$ in the lengths, we should then reduce the motion of the point to the case already considered, the initial velocity being $3t^2$, and the accelerating force $6t$. For we have already shown that a being the initial velocity, and g the accelerating force, the space described in the time t is $at + \frac{1}{2}gt^2$. Hence, $3t^2$ being the initial velocity, and $6t$ the accelerating force, the space in the time dt is $3t^2dt + 3t(dt)^2$, which is the same as AB after $(dt)^3$ is rejected. The velocity acquired is gt, and the whole velocity is, therefore, $a + gt$; or making the same substitutions $3t^2 + 6tdt$. This is the velocity at B, after the term $3(dt)^2$ is rejected. Again, the velocity being $3t^2 + 6t\,dt$, and the force $6t$, the space described in the time dt is $(3t^2 + 6t\,dt)dt + 3t(dt)^2$, or $3t^2dt + 9t(dt)^2$. This is what the space BC becomes after $7(dt)^3$ is rejected. The velocity acquired is $6t\,dt$; and the whole velocity is $3t^2 + 6t\,dt + 6t\,dt$, or $3t^2 + 12t\,dt$; which is the velocity at C after $12(dt)^2$ is rejected.

But as the terms involving $(dt)^2$ in the velocities, etc., cannot be rejected without error, the above supposition of a uniform force cannot be made. Nevertheless, as we may take dt so small that these terms shall be as small parts as we please of those which precede, the results of the erroneous and correct suppositions may be brought as near to equality as we please; hence we conclude, that though there is no force, which, continued uniformly, would preserve

the motion of the point A, so that OA should always be t^3 in inches, yet an interval of time may be taken so small, that the length actually described by A in that time, and the one which would be described if the force $6t$ were continued uniformly, shall have a ratio as near to equality as we please. Hence, on a principle similar to that by which we called $3t^2$ the velocity at A, though, in truth, no space, however small, is described with that velocity, we call $6t$ the accelerating force at A. And it must be observed that $6t$ is the differential coefficient of $3t^2$, or the coefficient of dt, in the development of $3(t+dt)^2$.

Generally, let the point move so that the length described in any time t is φt. Hence the length described at the end of the time $t+dt$ is $\varphi(t+dt)$, and that described in the interval dt is $\varphi(t+dt)-\varphi t$, or

$$\varphi' t\, dt + \varphi'' t\,\frac{(dt)^2}{2} + \varphi''' t\,\frac{(dt)^3}{2.3} + \text{etc.}$$

in which dt may be taken so small, that either of the first two terms shall contain the aggregate of all the rest, as often as we please. These two first terms are $\varphi' t\, dt + \frac{1}{2}\varphi'' t\,(dt)^2$, and represent the length described during dt, with a uniform velocity $\varphi' t$, and an accelerating force $\varphi'' t$. The interval dt may then generally be taken so small, that this supposition shall represent the motion during that interval as nearly as we please.

LIMITING RATIOS OF MAGNITUDES THAT INCREASE WITHOUT LIMIT.

We have hitherto considered the limiting ratio of quantities only as to their state of *decrease:* we now proceed to some cases in which the limiting ratio of different magnitudes which *increase* without limit is investigated.

It is easy to show that the increase of two magnitudes may cause a decrease of their ratio; so that, as the two increase without limit, their ratio may diminish without limit. The limit of any ratio may be found by rejecting any terms or aggregate of terms (Q) which are connected with another term (P) by the sign of addition or subtraction, provided that by increasing x, Q may be made as small a part of P as we please. For example, to find the limit of $\frac{x^2+2x+3}{2x^2+5x}$, when x is increased without limit. By increasing x we can, as will be shown immediately, cause $2x+3$ and $5x$ to be contained in x^2 and $2x^2$, as often as we please; rejecting these terms, we have $\frac{x^2}{2x^2}$, or $\frac{1}{2}$, for the limit.

The demonstration is as follows: Divide both numerator and denominator by x^2, which gives $1+\frac{2}{x}+\frac{3}{x^2}$, and $2+\frac{5}{x}$, for the numerator and denominator of a fraction equal in value to the one proposed. These can be brought as near as we please to 1 and 2 by making x sufficiently great, or $\frac{1}{x}$ sufficiently small; and, consequently, their ratio can be brought as near as we please to $\frac{1}{2}$.

We will now prove the following: That in any series of decreasing powers of x, any one term will, if x be taken sufficiently great, contain the aggregate of all which follow, as many times as we please. Take, for example,

$$ax^m + bx^{m-1} + cx^{m-2} + \ldots\ldots + px + q$$
$$+ \frac{r}{x} + \frac{s}{x^2} + \text{etc.}$$

The ratio of the several terms will not be altered if we divide the whole by x^m, which gives

$$a + \frac{b}{x} + \frac{c}{x^2} + \ldots\ldots + \frac{p}{x^{m-1}} + \frac{q}{x^m} + \frac{r}{x^{m+1}}$$
$$+ \frac{s}{x^{m+2}} + \text{etc.}$$

It has been shown that by taking $\frac{1}{x}$ sufficiently small, that is, by taking x sufficiently great, any term of this series may be made to contain the aggregate of the succeeding terms, as often as we please; which relation is not altered if we multiply every term by x^m, and so restore the original series.

It follows from this, that $\frac{(x+1)^m}{x^m}$ has unity for its limit when x is increased without limit. For $(x+1)^m$ is $x^m + mx^{m-1} +$ etc., in which x^m can be made as great as we please with respect to the rest of the series. Hence $\frac{(x+1)^m}{x^m} = 1 + \frac{mx^{m-1} + \text{etc.}}{x^m}$, the numerator of which last fraction decreases indefinitely as compared with its denominator.

In a similar way it may be shown that the limit of $\frac{x^m}{(x+1)^{m+1} - x^{m+1}}$, when x is increased, is $\frac{1}{m+1}$. For since $(x+1)^{m+1} = x^{m+1} + (m+1)x^m + \frac{1}{2}(m+1)mx^{m-1}$ + etc., this fraction is

$$\frac{x^m}{(m+1)x^m + \frac{1}{2}(m+1)mx^{m-1} + \text{etc.}}$$

in which the first term of the denominator may be made to contain all the rest as often as we please; that is, if the fraction be written thus, $\frac{x^m}{(m+1)x^m + \text{A}}$, A can be made as small a part of $(m+1)x^m$ as we

please. Hence this fraction can, by a sufficient increase of x, be brought as near as we please to $\frac{x^m}{(m+1)x^m}$, or $\frac{1}{m+1}$.

A similar proposition may be shown of the fraction $\frac{(x+b)^m}{(x+a)^{m+1}-x^{m+1}}$, which may be immediately reduced to the form $\frac{x^m+\mathrm{B}}{(m+1)\,a\,x^m+\mathrm{A}}$, where x may be taken so great that x^m shall contain A and B any number of times.

We will now consider the sums of x terms of the following series, each of which may evidently be made as great as we please, by taking a sufficient number of its terms,

$$
\begin{array}{lr}
1 + 2 + 3 + 4 + \ldots\ldots\ldots + x-1 + x & (1)\\
1^2 + 2^2 + 3^2 + 4^2 + \ldots\ldots\ldots + (x-1)^2 + x^2 & (2)\\
1^3 + 2^3 + 3^3 + 4^3 + \ldots\ldots\ldots + (x-1)^3 + x^3 & (3)\\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \\
1^m + 2^m + 3^m + 4^m + \ldots\ldots\ldots + (x-1)^m + x^m & (m)
\end{array}
$$

We propose to inquire what is the limiting ratio of any one of these series to the last term of the succeeding one; that is, to what do the ratios of $(1+2+\ldots\ldots+x)$ to x^2, of $(1^2+2^2\ldots\ldots+x^2)$ to x^3, etc., approach, when x is increased without limit.

To give an idea of the method of increase of these series, we shall first show that x may be taken so great, that the last term of each series shall be as small a part as we please of the sum of all those which precede. To simplify the symbols, let us take the third series $1^3+2^3+\ldots\ldots+x^3$, in which we are to show that x^3 may be made less than any given part,

say one thousandth, of the sum of those which precede, or of $1^3 + 2^3 \ldots . + (x-1)^3$.

First, x may be taken so great that x^3 and $(x-1000)^3$ shall have a ratio as near to equality as we please. For the ratio of these quantities being the same as that of 1 to $\left(1 - \frac{1000}{x}\right)^3$, and $\frac{1000}{x}$ being as small as we please if x may be as great as we please, it follows that $1 - \frac{1000}{x}$, and, consequently, $\left(1 - \frac{1000}{x}\right)^3$ may be made as near to unity as we please, or the ratio of 1 to $\left(1 - \frac{1000}{x}\right)^3$, may be brought as near as we please to that of 1 to 1, or a ratio of equality. But this ratio is that of x^3 to $(x-1000)^3$. Similarly the ratios of x^3 to $(x-999)^3$, of x^3 to $(x-998)^3$, etc., up to the ratio of x^3 to $(x-1)^3$ may be made as near as we please to ratios of equality; there being one thousand in all. If, then, $(x-1)^3 = \alpha x^3$, $(x-2)^3 = \beta x^3$, etc., up to $(x-1000)^3 = \omega x^3$, x can be taken so great that each of the fractions α, β, etc., shall be as near to unity, or $\alpha + \beta + \ldots . + \omega$ as near* to 1000 as we please. Hence $\frac{1}{\alpha + \beta + \ldots + \omega}$ which is

$$\frac{x^3}{\alpha x^3 + \beta x^3 + \ldots . + \omega x^3}, \text{ or}$$

$$\frac{x^3}{(x-1)^3 + (x-2)^3 + \ldots + (x-1000)^3},$$

*Observe that this conclusion depends upon the *number* of quantities α, β, etc., being *determinate.* If there be *ten* quantities, each of which can be brought as near to unity as we please, their sum can be brought as near to 10 as we please; for, take any fraction A, and make each of those quantities differ from unity by less than the tenth part of A, then will the sum differ from 10 by less than A. This argument fails, if the number of quantities be unlimited.

can be brought as near to $\frac{1}{1000}$ as we please; and by the same reasoning, the fraction

$$\frac{x^3}{(x-1)^3+\ldots\ldots+(x-1001)^3}$$

may be brought as near to $\frac{1}{1001}$ as we please; that is, may be made less than $\frac{1}{1000}$. Still more then may

$$\frac{x^3}{(x-1)^3+\ldots\ldots+(x-1001)^3+\ldots\ldots+2^3+1^3}$$

be made less than $\frac{1}{1000}$, or x^3 may be less than the thousandth part of the sum of all the preceding terms.

In the same way it may be shown that a term may be taken in any one of the series, which shall be less than any given part of the sum of all the preceding terms. It is also true that the difference of any two succeeding terms may be made as small a part of either as we please. For $(x+1)^m-x^m$, when developed, will only contain exponents less than m, being $mx^{m-1}+m\,\frac{m-1}{2}\,x^{m-2}+$ etc.; and we have shown (page 66) that the sum of such a series may be made less than any given part of x^m. It is also evident that, whatever number of terms we may sum, if a sufficient number of succeeding terms be taken, the sum of the latter shall exceed that of the former in any ratio we please.

Let there be a series of fractions

$$\frac{a}{pa+b},\ \frac{a'}{pa'+b'},\ \frac{a''}{pa''+b''},\ \text{etc.},$$

in which a, a', etc., b, b', etc., increase without limit; but in which the ratio of b to a, b' to a', etc., diminishes without limit. If it be allowable to begin by

supposing b as small as we please with respect to a, or $\frac{b}{a}$ as small as we please, the first, and all the succeeding fractions, will be as near as we please to $\frac{1}{p}$, which is evident from the equations

$$\frac{a}{pa+b}=\frac{1}{p+\frac{b}{a}},\ \frac{a'}{pa'+b'}=\frac{1}{p+\frac{b'}{a'}},\ \text{etc.}$$

Form a new fraction by summing the numerators and denominators of the preceding, such as

$$\frac{a+a'+a''+\text{etc.}}{p(a+a'+a''+\text{etc.})+b+b'+b''+\text{etc.}},$$

the *etc.* extending to any given number of terms. This may also be brought as near to $\frac{1}{p}$ as we please. For this fraction is the same as

$$1 \text{ divided by } p+\frac{b+b'+\text{etc.}}{a+a'+\text{etc.}};$$

and it can be shown* that

$$\frac{b+b'+\text{etc.}}{a+a'+\text{etc.}}$$

must lie between the least and greatest of the fractions $\frac{b}{a}$, $\frac{b'}{a'}$, etc. If, then, each of these latter fractions can be made as small as we please, so also can

$$\frac{b+b'+\text{etc.}}{a+a'+\text{etc.}}.$$

No difference will be made in this result, if we use the following fraction,

$$\frac{\text{A}+(a+a'+a''+\text{etc.})}{\text{B}+p(a+a'+a''+\text{etc.})+b+b'+b''+\text{etc.}} \qquad (1)$$

* See *Study of Mathematics* (Reprint Edition, Chicago: The Open Court Publishing Co.), page 270.

A and B being given quantities; provided that we can take a number of the original fractions sufficient to make $a+a'+a''+\text{etc.}$, as great as we please, compared with A and B. This will appear on dividing the numerator and denominator of (1) by $a+a'+a''+\text{etc.}$

Let the fractions be

$$\frac{(x+1)^3}{(x+1)^4-x^4}, \quad \frac{(x+2)^3}{(x+2)^4-(x+1)^4},$$

$$\frac{(x+3)^3}{(x+3)^4-(x+2)^4}, \text{ etc.}$$

The first of which, or $\frac{(x+1)^3}{4x^3+\text{etc.}}$ may, as we have shown, be within any given difference of $\frac{1}{4}$, and the others still nearer, by taking a value of x sufficiently great. Let us suppose each of these fractions to be within $\frac{1}{100000}$ of $\frac{1}{4}$. The fraction formed by summing the numerators and denominators of these fractions (n in number) will be within the same degree of nearness to $\frac{1}{4}$. But this is

$$\frac{(x+1)^3+(x+2)^3+\ldots\ldots\ldots+(x+n)^3}{(x+n)^4-x^4} \qquad (2)$$

all the terms of the denominator disappearing, except two from the first and last. If, then, we add x^4 to the denominator, and $1^3+2^3+3^3\ldots\ldots+x^3$ to the numerator, we can still take n so great that $(x+1)^3+\ldots\ldots+(x+n)^3$ shall contain $1^3+\ldots+x^3$ as often as we please, and that $(x+n)^4-x^4$ shall contain x^4 in the same manner. To prove the latter, observe that the ratio of $(x+n)^4-x^4$ to x^4 being $\left(1+\frac{n}{x}\right)^4$, can be made as great as we please, if it be permitted

to take for n a number containing x as often as we please. Hence, by the preceding reasoning, the fraction, with its numerator and denominator thus increased, or

$$\frac{1^3+2^3+3^3+\ldots.+x^3+(x+1)^3+\ldots+(x+n)^3}{(x+n)^4} \quad (3)$$

may be brought to lie within the same degree of nearness to $\frac{1}{4}$ as (2); and since this degree of nearness could be named at pleasure, it follows that (3) can be brought as near to $\frac{1}{4}$ as we please. Hence the limit of the ratio of $(1^3+2^3+\ldots.+x^3)$ to x^4, as x is increased without limit, is $\frac{1}{4}$; and, in a similar manner, it may be proved that the limit of the ratio of $(1^m+2^m+\ldots.+x^m)$ to x^{m+1} is the same as that of $\dfrac{(x+1)^m}{(x+1)^{m+1}-x^{m+1}}$ or $\dfrac{1}{m+1}$.

This result will be of use when we come to the first principles of the integral calculus. It may also be noticed that the limits of the ratios which $x\,\dfrac{x-1}{2}$, $x\,\dfrac{x-1}{2}\,\dfrac{x-2}{3}$, etc., bear to x^2, x^3, etc., are severally $\dfrac{1}{2}$, $\dfrac{1}{2.3}$, etc.; the limit being that to which the ratios approximate as x increases without limit. For $x\,\dfrac{x-1}{2} \div x^2 = \dfrac{x-1}{2x}$, $x\,\dfrac{x-1}{2}\,\dfrac{x-2}{3} \div x^3 = \dfrac{x-1}{2x}\,\dfrac{x-2}{3x}$, etc., and the limits of $\dfrac{x-1}{x}$, $\dfrac{x-2}{x}$, are severally equal to unity.

We now resume the elementary principles of the Differential Calculus.

RECAPITULATION OF RESULTS.

The following is a recapitulation of the principal results which have hitherto been noticed in the general theory of functions:

(1) That if in the equation $y = \varphi(x)$, the variable x receives an increment dx, y is increased by the series

$$\varphi' x\, dx + \varphi'' x \frac{(dx)^2}{2} + \varphi'' x \frac{(dx)^3}{2.3} + \text{etc.}$$

(2) That $\varphi'' x$ is derived in the same manner from $\varphi' x$, that $\varphi' x$ is from φx; viz., that in like manner as $\varphi' x$ is the coefficient of dx in the development of $\varphi(x + dx)$, so $\varphi'' x$ is the coefficient of dx in the development of $\varphi'(x + dx)$; similarly $\varphi''' x$ is the coefficient of dx in the development of $\varphi''(x + dx)$, and so on.

(3) That $\varphi' x$ is the limit of $\frac{dy}{dx}$, or the quantity to which the latter will approach, and to which it may be brought as near as we please, when dx is diminished. It is called the differential coefficient of y.

(4) That in every case which occurs in practice, dx may be taken so small, that any term of the series above written may be made to contain the aggregate of those which follow, as often as we please; whence, though $\varphi' x\, dx$ is not the actual increment produced by changing x into $x + dx$ in the function φx, yet, by taking dx sufficiently small, it may be brought as near as we please to a ratio of equality with the actual increment.

APPROXIMATIONS.

The last of the above-mentioned principles is of the greatest utility, since, by means of it, $\varphi' x\, dx$ may

be made as nearly as we please the actual increment; and it will generally happen in practice, that $\varphi' x\,dx$ may be used for the increment of φx without sensible error; that is, if in φx, x be changed into $x + dx$, dx being very small, φx is changed into $\varphi x + \varphi' x\,dx$, very nearly. Suppose that x being the correct value of the variable, $x + h$ and $x + k$ have been successively substituted for it, or the errors h and k have been committed in the valuation of x, h and k being very small. Hence $\varphi(x + h)$ and $\varphi(x + k)$ will be erroneously used for φx. But these are nearly $\varphi x + \varphi' x\,h$ and $\varphi x + \varphi' x\,k$, and the errors committed in taking φx are $\varphi' x\,h$ and $\varphi' x\,k$, very nearly. These last are in the proportion of h to k, and hence results a proposition of the utmost importance in every practical application of mathematics, viz., that if two different, but small, errors be committed in the valuation of any quantity, the errors arising therefrom at the end of any process, in which both the supposed values of x are successively adopted, are very nearly in the proportion of the errors committed at the beginning. For example, let there be a right-angled triangle, whose base is 3, and whose other side should be 4, so that the hypothenuse should be $\sqrt{3^2 + 4^2}$ or 5. But suppose that the other side has been twice erroneously measured, the first measurement giving 4·001, and the second 4·002, the errors being ·001 and ·002. The two values of the hypothenuse thus obtained are

$$\sqrt{3^2 + 4{\cdot}001^2}, \text{ or } \sqrt{25{\cdot}008001},$$
$$\text{and } \sqrt{3^2 + 4{\cdot}002^2}, \text{ or } \sqrt{25{\cdot}016004},$$

which are very nearly 5·0008 and 5·0016. The errors of the hypothenuse are then ·0008 and ·0016 nearly; and these last are in the proportion of ·001 and ·002.

It also follows, that if x increase by successive equal steps, any function of x will, for a few steps, increase so nearly in the same manner, that the supposition of such an increase will not be materially wrong. For, if h, $2h$, $3h$, etc., be successive small increments given to x, the successive increments of φx will be $\varphi' x\, h$, $\varphi' x\, 2h$, $\varphi' x\, 3h$, etc. nearly; which being proportional to h, $2h$, $3h$, etc., the increase of the function is nearly doubled, trebled, etc., if the increase of x be doubled, trebled, etc.

This result may be rendered conspicuous by reference to any astronomical ephemeris, in which the positions of a heavenly body are given from day to day. The intervals of time at which the positions are given differ by 24 hours, or nearly $\frac{1}{365}$th part of the whole year. And even for this interval, though it can hardly be called *small* in an astronomical point of view, the increments or decrements will be found so nearly the same for four or five days together, as to enable the student to form an idea how much more near they would be to equality, if the interval had been less, say one hour instead of twenty-four. For example, the sun's longitude on the following days at noon is written underneath, with the increments from day to day.

1834 September	Sun's longitude at noon.	Increments.	Proportion which the differences of the increments bear to the whole increments.
1st	158° 30′ 35″		
		58′ 9″	
2nd	159 28 44		$\frac{3}{3489}$
		58 12	
3rd	160 26 56		$\frac{1}{3492}$
		58 13	
4th	161 25 9		$\frac{1}{3493}$
		58 14	
5th	162 23 23		

The sun's longitude is a function of the time; that is, the number of years and days from a given epoch being given, and called x, the sun's longitude can be

found by an algebraical expression which may be called φx. If we date from the first of January, 1834, x is $\cdot 666$, which is the decimal part of a year between the first days of January and September. The increment is one day, or nearly $\cdot 0027$ of a year. Here x is successively made equal to $\cdot 666$, $\cdot 666 + \cdot 0027$, $\cdot 666 + 2 \times \cdot 0027$, etc.; and the intervals of the corresponding values of φx, if we consider only minutes, are the same; but if we take in the seconds, they differ from one another, though only by very small parts of themselves, as the last column shows.

SOLUTION OF EQUATIONS.

This property is also used* in finding logarithms intermediate to those given in the tables; and may be applied to find a nearer solution to an equation, than one already found. For example, suppose it required to find the value of x in the equation $\varphi x = 0$, a being a near approximation to the required value. Let $a + h$ be the real value, in which h will be a small quantity. It follows that $\varphi(a + h) = 0$, or, which is nearly true, $\varphi a + \varphi' a\, h = 0$. Hence the real value of h is nearly $-\dfrac{\varphi a}{\varphi' a}$, or the value $a - \dfrac{\varphi a}{\varphi' a}$ is a nearer approximation to the value of x. For example, let $x^2 + x - 4 = 0$ be the equation. Here $\varphi x = x^2 + x - 4$, and $\varphi(x + h) = (x + h)^2 + x + h - 4 = x^2 + x - 4 + (2x + 1)h + h^2$; so that $\varphi' x = 2x + 1$. A near value of x is $1 \cdot 57$; let this be a. Then $\varphi a = \cdot 0349$, and $\varphi' a = 4 \cdot 14$. Hence $-\dfrac{\varphi a}{\varphi' a} = - \cdot 00843$. Hence $1 \cdot 57 - \cdot 00843$, or $1 \cdot 56157$, is a nearer value of x. If

*See *Study of Mathematics* (Reprint Edition, Chicago: The Open Court Publishing Co., 1898), page 169 et seq.

we proceed in the same way with 1·5616, we shall find a still nearer value of x, viz., 1·561553. We have here chosen an equation of the second degree, in order that the student may be able to verify the result in the common way; it is, however, obvious that the same method may be applied to equations of higher degrees, and even to those which are not to be treated by common algebraical method, such as $\tan x = ax$.

PARTIAL AND TOTAL DIFFERENTIALS.

We have already observed, that in a function of more quantities than one, those only are mentioned which are considered as variable; so that all which we have said upon functions of one variable, applies equally to functions of several variables, so far as a change in one only is concerned. Take for example $x^2y + 2xy^3$. If x be changed into $x + dx$, y remaining the same, this function is increased by $2xy\,dx + 2y^3dx$ + etc., in which, as in page 29, no terms are contained in the *etc.* except those which, by diminishing dx, can be made to bear as small a proportion as we please to the first terms. Again, if y be changed into $y + dy$, x remaining the same, the function receives the increment $x^2dy + 6xy^2dy$ + etc.; and if x be changed into $x + dx$, y being at the same time changed into $y + dy$, the increment of the function is $(2xy + 2y^3)dx + (x^2 + 6xy^2)\,dy$ + etc. If, then, $u = x^2y + 2xy^3$, and du denote the increment of u, we have the three following equations, answering to the various suppositions above mentioned,

(1) when x only varies,

$$du = (2xy + 2y^3)\,dx + \text{etc.}$$

(2) when y only varies,

$$du = (x^2 + 6xy^2)\,dy + \text{etc.}$$

(3) when both x and y vary,

$$du = (2xy + 2y^3)\,dx + (x^2 + 6xy^2)\,dy + \text{etc.}$$

in which, however, it must be remembered, that du does not stand for the same thing in any two of the three equations: it is true that it always represents an increment of u, but as far as we have yet gone, we have used it indifferently, whether the increment of u was the result of a change in x only, or y only, or both together.

To distinguish the different increments of u, we must therefore seek an additional notation, which, without sacrificing the du that serves to remind us that it was u which received an increment, may also point out from what supposition the increment arose. For this purpose we might use $d_x u$ and $d_y u$, and $d_{x,y} u$, to distinguish the three; and this will appear to the learner more simple than the one in common use, which we shall proceed to explain. We must, however, remind the student, that though in matters of reasoning, he has a right to expect a solution of every difficulty, in all that relates to notation, he must trust entirely to his instructor; since he cannot judge between the convenience or inconvenience of two symbols without a degree of experience which he evidently cannot have had. Instead of the notation above described, the increments arising from a change in x and y are severally denoted by $\frac{du}{dx}dx$ and $\frac{du}{dy}dy$, on the following principle: If there be a number of results obtained by the same species of process, but on different suppositions with regard to the quantities

used ; if, for example, p be derived from some supposition with regard to a, in the same manner as are q and r with regard to b and c, and if it be inconvenient and unsymmetrical to use separate letters p, q, and r, for the three results, they may be distinguished by using the same letter p for all, and writing the three results thus, $\frac{p}{a}a$, $\frac{p}{b}b$, $\frac{p}{c}c$. Each of these, in common algebra, is equal to p, but the letter p does not stand for the same thing in the three expressions. The first is the p, so to speak, which belongs to a, the second that which belongs to b, the third that which belongs to c. Therefore the numerator of each of the fractions $\frac{p}{a}$, $\frac{p}{b}$, and $\frac{p}{c}$, must never be separated from its denominator, because the value of the former depends, in part, upon the latter; and one p cannot be distinguished from another without its denominator. The numerator by itself only indicates what operation is to be performed, and on what quantity; the denominator shows what quantity is to be made use of in performing it. Neither are we allowed to say that $\frac{p}{a}$ divided by $\frac{p}{b}$ is $\frac{b}{a}$; for this supposes that p means the same thing in both quantities.

In the expressions $\frac{du}{dx}dx$, and $\frac{du}{dy}dy$, each denotes that u has received an increment; but the first points out that x, and the second that y, was supposed to increase, in order to produce that increment; while du by itself, or sometimes $d.u$, is employed to express the increment derived from both suppositions at once. And since, as we have already remarked, it is not the ratios of the increments themselves, but the limits of those ratios, which are the objects of investigation in

the Differential Calculus, here, as in page 28, $\frac{du}{dx}dx$, and $\frac{du}{dy}dy$, are generally considered as representing those terms which are of use in obtaining the limiting ratios, and do not include those terms, which, from their containing higher powers of dx or dy than the first, may be made as small as we please with respect to dx or dy. Hence in the example just given, where $u = x^2y + 2xy^3$, we have

$$\frac{du}{dx}dx = (2xy + 2y^3)\,dx, \qquad \text{or } \frac{du}{dx} = 2xy + 2y^3$$

$$\frac{du}{dy}dy = (x^2 + 6xy^2)\,dy, \qquad \text{or } \frac{du}{dy} = x^2 + 6xy^2$$

$$du \text{ or } d.u = \frac{du}{dx}dx + \frac{du}{dy}dy.$$

The last equation gives a striking illustration of the method of notation. Treated according to the common rules of algebra, it is $du = du + du$, which is absurd, but which appears rational when we recollect that the second du arises from a change in x only, the third from a change in y only, and the first from a change in both. The same equation may be proved to be generally true for all functions of x and y, if we bear in mind that no term is retained, or need be retained, as far as the limit is concerned, which, when dx or dy is diminished, diminishes without limit as compared with them. In using $\frac{du}{dx}$ and $\frac{du}{dy}$ as differential coefficients of u with respect to x and y, the objection (page 27) against considering these as the limits of the ratios, and not the ratios themselves, does not hold, since the numerator is not to be separated from its denominator.

Let u be a function of x and y, represented* by $\varphi(x, y)$. It is indifferent whether x and y be changed at once into $x + dx$ and $y + dy$, or whether x be first changed into $x + dx$, and y be changed into $y + dy$ in the result. Thus, $x^2y + y^3$ will become $(x + dx)^2 (y + dy) + (y + dy)^3$ in either case. If x be changed into $x + dx$, u becomes $u + u'dx +$ etc., (where u' is what we have called the differential coefficient of u with respect to x, and is itself a function of x and y; and the corresponding increment of u is $u'dx +$ etc.) If in this result y be changed into $y + dy$, u will assume the form $u + u_{,}dy +$ etc., where $u_{,}$ is the differential coefficient of u with respect to y; and the increment which u receives will be $u_{,}dy +$ etc. Again, when y is changed into $y + dy$, u', which is a function of x and y, will assume the form $u' + p\,dy +$ etc.; and $u + u'dx +$ etc. becomes $u + u_{,}dy +$ etc. $+ (u' + p\,dy +$ etc.$)\,dx +$ etc., or $u + u_{,}dy + u'dx + p\,dx\,dy +$ etc., in which the term $p\,dx\,dy$ is useless in finding the limit. For since dy can be made as small as we please, $p\,dx\,dy$ can be made as small a part of $p\,dx$ as we please, and therefore can be made as small a part of dx as we please. Hence on the three suppositions already made, we have the following results:

(1) when x only is changed into $x + dx$,	u receives the increment	$u'dx +$ etc.
(2) when y only is changed into $y + dy$,		$u_{,}dy +$ etc.
(3) when x becomes $x + dx$ and y becomes $y + dy$ at once,		$u'dx + u_{,}dy +$ etc.

*The symbol $\phi(x, y)$ must not be confounded with $\phi(xy)$. The former represents any function of x and y; the latter a function in which x and y only enter so far as they are contained in their product. The second is therefore a particular case of the first; but the first is not necessarily represented by

the *etc.* in each case containing those terms only which can be made as small as we please, with respect to the preceding terms. In the language of Leibnitz, we should say that if x and y receive infinitely small increments, the sum of the infinitely small increments of u obtained by making these changes separately, is equal to the infinitely small increment obtained by making them both at once. As before, we may correct this inaccurate method of speaking. The several increments in (1), (2), and (3), may be expressed by $u'dx + \mathrm{P}$, $u_{\prime}dy + \mathrm{Q}$, and $u'dx + u_{\prime}dy + \mathrm{R}$; where P, Q, and R can be made such parts of dx or dy as we please, by taking dx or dy sufficiently small. The sum of the two first is $u'dx + u_{\prime}dy + \mathrm{P} + \mathrm{Q}$, which differs from the third by $\mathrm{P} + \mathrm{Q} - \mathrm{R}$; which, since each of its terms can be made as small a part of dx or dy as we please, can itself be made less than any given part of dx or dy.

This theorem is not confined to functions of two variables only, but may be extended to those of any number whatever. Thus, if z be a function of p, q, r, and s, we have

$$d.z \text{ or } dz = \frac{dz}{dp}dp + \frac{dz}{dq}dq + \frac{dz}{dr}dr + \frac{dz}{ds}ds + \text{etc.}$$

in which $\frac{dz}{dp}dp +$ etc. is the increment which a change in p *only* gives to z, and so on. The *etc.* is the representative of an infinite series of terms, the aggregate of which diminishes continually with respect to dp, dq, etc., as the latter are diminished, and which, there-

the second. For example, take the function $xy + \sin xy$, which, though it contains both x and y, yet can only be altered by such a change in x and y as will alter their product, and if the product be called p, will be $p + \sin p$. This may properly be represented by $\phi(xy)$; whereas $x + xy^2$ cannot be represented in the same way, since other functions besides the product are contained in it.

fore, has no effect on the *limit* of the ratio of $d.z$ to any other quantity.

PRACTICAL APPLICATION OF THE PRECEDING THEOREM.

We proceed to an important practical use of this theorem. If the increments dp, dq, etc., be small, this last-mentioned equation, (the terms included in the *etc.* being omitted,) though not actually true, is sufficiently near the truth for all practical purposes; which renders the proposition, from its simplicity, of the highest use in the applications of mathematics. For if any result be obtained from a set of *data*, no one of which is exactly correct, the error in the result would be a very complicated function of the errors in the *data*, if the latter were considerable. When they are small, the error in the results is very nearly the sum of the errors which would arise from the error in each *datum*, if all the others were correct. For if p, q, r, and s, are the *presumed* values of the *data*, which give a certain value z to the function required to be found; and if $p+dp$, $q+dq$, etc., be the *correct* values of the *data*, the correction of the function z will be very nearly made, if z be increased by $\frac{dz}{dp}dp+\frac{dz}{dq}dq+\frac{dz}{dr}dr+\frac{dz}{ds}ds$, being the sum of terms which would arise from each separate error, if each were made in turn by itself.

For example: A transit instrument is a telescope mounted on an axis, so as to move in the plane of the meridian only, that is, the line joining the centres of the two glasses ought, if the telescope be moved, to pass successively through the zenith and the pole. Hence can be determined the exact time, as shown by a clock, at which any star passes a vertical thread,

fixed inside the telescope so as apparently to cut the field of view exactly in half, which thread will always cover a part of the meridian, if the telescope be correctly adjusted. In trying to do this, three errors may, and generally will be committed, in some small degree. (1) The axis of the telescope may not be exactly level; (2) the ends of the same axis may not be exactly east and west; (3) the line which joins the centres of the two glasses, instead of being perpendicular to the axis of the telescope, may be inclined to it. If each of these errors were considerable, and the time at which a star passed the thread were observed, the calculation of the time at which the same star passes the real meridian would require complicated formulæ, and be a work of much labor. But if the errors exist in small quantities only, the calculation is very much simplified by the preceding principle. For, suppose only the first error to exist, and calculate the corresponding error in the time of passing the thread. Next suppose only the second error, and then only the third to exist, and calculate the effect of each separately, all which may be done by simple formulæ. The effect of all the errors will then be the sum of the effects of each separate error, at least with sufficient accuracy for practical purposes. The formulæ employed, like the equations in page 28, are not actually true in any case, but approach more near to the truth as the errors are diminished.

RULES FOR DIFFERENTIATION.

In order to give the student an opportunity of exercising himself in the principles laid down, we will so far anticipate the treatises on the Differential Calculus as to give the results of all the common rules

for differentiation; that is, assuming y to stand for various functions of x, we find the increment of y arising from an increment in the value of x, or rather, that term of the increment which contains the first power of dx. This term, in theory, is the only one on which the *limit* of the ratio of the increments depends; in practice, it is sufficiently near to the real increment of y, if the increment of x be small.

(1) $y = x^m$, where m is either whole or fractional, positive or negative; then $dy = mx^{m-1}dx$. Thus the increment of $x^{\frac{2}{3}}$ or the first term of $(x+dx)^{\frac{2}{3}} - x^{\frac{2}{3}}$ is $\frac{2}{3}x^{\frac{2}{3}-1}dx$, or $\frac{2dx}{3x^{\frac{1}{3}}}$. Again, if $y = x^8$, $dy = 8x^7 dx$. When the exponent is negative, or when $y = \frac{1}{x^m}$, $dy = -\frac{mdx}{x^{m+1}}$, or when $y = x^{-m}$, $dy = -mx^{-m-1}dx$, which is according to the rule. The negative sign indicates that an increase in x decreases the value of y; which, in this case, is evident.

(2) $y = a^x$. Here $dy = a^x \log a\, dx$ where the logarithm (as is always the case in analysis, except where the contrary is specially mentioned) is the Naperian or hyperbolic logarithm. When a is the base of these logarithms, that is when $a = 2\cdot7182818 = \varepsilon$, or when $y = \varepsilon^x$, $dy = \varepsilon^x dx$.

(3) $y = \log x$ (the Naperian logarithm). Here $dy = \frac{dx}{x}$. If $y =$ common $\log x$, $dy = \cdot4342944\,\frac{dx}{x}$.

(4) $y = \sin x$, $dy = \cos x\, dx$; $y = \cos x$, $dy = -\sin x\, dx$; $y = \tan x$, $dy = \frac{dx}{\cos^2 x}$.

ILLUSTRATION OF THE PRECEDING FORMULÆ.

At the risk of being tedious to some readers, we will proceed to illustrate these formulæ by examples

from the tables of logarithms and sines. Let $y =$ common $\log x$. If x be changed into $x + dx$, the real increment of y is

$$\cdot 4342944 \left(\frac{dx}{x} - \frac{1}{2} \frac{(dx)^2}{x^2} + \frac{1}{3} \frac{(dx)^3}{x^3} - \text{etc.} \right),$$

in which the law of continuation is evident. The corresponding series for Naperian logarithms is to be found in page 20. From the first term of this the limit of the ratio of dy to dx can be found; and if dx be small, this will represent the increment with sufficient accuracy. Let $x = 1000$, whence $y =$ common $\log 1000 = 3$; and let $dx = 1$, or let it be required to find the common logarithm of $1000 + 1$, or 1001. The first term of the series is therefore $\cdot 4342944 \times \frac{1}{1000}$, or $\cdot 0004343$, taking seven decimal places only. Hence $\log 1001 = \log 1000 + \cdot 0004343$ or $3 \cdot 0004343$ nearly. The tables give $3 \cdot 0004341$, differing from the former only in the 7th place of decimals.

Again, let $y = \sin x$; from which, by page 20, as before, if x be increased by dx, $\sin x$ is increased by $\cos x \, dx - \frac{1}{2} \sin x \, (dx)^2 -$ etc., of which we take only the first term. Let $x = 16°$, in which case $\sin x = \cdot 2756374$, and $\cos x = \cdot 9612617$. Let $dx = 1'$, or, as it is represented in analysis, where the angular unit is that angle whose arc is equal to the radius*, $\frac{60}{206265}$. Hence $\sin 16° \, 1' = \sin 16° + \cdot 9612617 \times \frac{60}{206265} = \cdot 2756374 + \cdot 0002797 = \cdot 2759171$, nearly. The tables give $\cdot 2759170$. These examples may serve to show how nearly the real ratio of two increments approaches to their limit, when the increments themselves are small.

* See *Study of Mathematics* (Chicago: The Open Court Pub. Co.), page 273 et seq.

DIFFERENTIAL COEFFICIENTS OF DIFFERENTIAL COEFFICIENTS.

When the differential coefficient of a function of x has been found, the result, being a function of x, may be also differentiated, which gives the differential coefficient of the differential coefficient, or, as it is called, the *second* differential coefficient. Similarly the differential coefficient of the second differential coefficient is called the third differential coefficient, and so on. We have already had occasion to notice these successive differential coefficients in page 22, where it appears that $\varphi' x$ being the first differential coefficient of φx, $\varphi'' x$ is the coefficient of h in the development $\varphi'(x+h)$, and is therefore the differential coefficient of $\varphi' x$, or what we have called the second differential coefficient of φx. Similarly $\varphi''' x$ is the third differential coefficient of φx. If we were strictly to adhere to our system of notation, we should denote the several differential coefficients of φx or y by

$$\frac{dy}{dx} \qquad \frac{d\,.\,\frac{dy}{dx}}{dx} \qquad \frac{d.\,\frac{d\,.\,\frac{dy}{dx}}{dx}}{dx} \text{ etc.}$$

In order to avoid so cumbrous a system of notation, the following symbols are usually preferred,

$$\frac{dy}{dx}\ \frac{d^2y}{dx^2}\ \frac{d^3y}{dx^3}, \text{ etc.}$$

CALCULUS OF FINITE DIFFERENCES. SUCCESSIVE DIFFERENTIATION.

We proceed to explain the manner in which this notation is connected with our previous ideas on the subject.

When in any function of x, an increase is given to x, which is not supposed to be as small as we please, it is usual to denote it by Δx instead of dx, and the corresponding increment of y or φx, by Δy or $\Delta\varphi x$, instead of dy or $d\varphi x$. The symbol Δx is called the *difference* of x, being the difference between the value of the variable x, before and after its increase.

Let x increase at successive steps by the same difference; that is, let a variable, whose first value is x, successively become $x+\Delta x$, $x+2\Delta x$, $x+3\Delta x$, etc., and let the successive values of φx corresponding to these values of x be y, y_1, y_2, y_3, etc.; that is, φx is called y, $\varphi(x+\Delta x)$ is y_1, $\varphi(x+2\Delta x)$ is y_2, etc., and, generally, $\varphi(x+m\Delta x)$ is y_m. Then, by our previous definition y_1-y is Δy, y_2-y_1 is Δy_1, y_3-y_2 is Δy_2, etc., the letter Δ before a quantity always denoting the increment it would receive if $x+\Delta x$ were substituted for x. Thus y_3 or $\varphi(x+3\Delta x)$ becomes $\varphi(x+\Delta x+3\Delta x)$, or $\varphi(x+4\Delta x)$, when x is changed into $x+\Delta x$, and receives the increment $\varphi(x+4\Delta x)-\varphi(x+3\Delta x)$, or y_4-y_3. If y be a function which decreases when x is increased, y_1-y, or Δy is negative.

It must be observed, as in page 26, that Δx does not depend upon x, because x occurs in it; the symbol merely signifies an increment given to x, which increment is not necessarily dependent upon the value of x. For instance, in the present case we suppose it a given quantity; that is, when $x+\Delta x$ is changed into $x+\Delta x+\Delta x$, or $x+2\Delta x$, x is changed, and Δx is not.

In this way we get the two first of the columns underneath, in which each term of the *second* column is formed by subtracting the term which immediately precedes it in the first column from the one which im-

mediately follows. Thus Δy is $y_1 - y$, Δy_1 is $y_2 - y_1$, etc.

$$
\begin{array}{ll|l|l|l|l}
\varphi(x) & y & \Delta y & \Delta^2 y & \Delta^3 y & \Delta^4 y \\
\varphi(x+\Delta x)\ldots & y_1 & \Delta y_1 & \Delta^2 y_1 & \Delta^3 y_1 & \\
\varphi(x+2\Delta x)\ldots & y_2 & \Delta y_2 & \Delta^2 y_2 & & \\
\varphi(x+3\Delta x)\ldots & y_3 & \Delta y_3 & & & \\
\varphi(x+4\Delta x)\ldots & y_4 & & & & \\
\text{etc.} & & & & &
\end{array}
$$

In the first column is to be found a series of successive values of the same function φx, that is, it contains terms produced by substituting successively in φx the quantities x, $x + \Delta x$, $x + 2\Delta x$, etc., instead of x. The second column contains the successive values of another function $\varphi(x + \Delta x) - \varphi x$, or $\Delta \varphi x$, made by the same substitutions; if, for example, we substitute $x + 2\Delta x$ for x, we obtain $\varphi(x + 3\Delta x) - \varphi(x + 2\Delta x)$, or $y_3 - y_2$, or Δy_2. If, then, we form the successive differences of the terms in the second column, we obtain a new series, which we might call the differences of the differences of the first column, but which are called the *second differences* of the first column. And as we have denoted the operation which deduces the second column from the first by Δ, so that which deduces the third from the second may be denoted by $\Delta\Delta$, which is abbreviated into Δ^2. Hence as $y_1 - y$ was written Δy, $\Delta y_1 - \Delta y$ is written $\Delta\Delta y$, or $\Delta^2 y$. And the student must recollect, that in like manner as Δ is not the symbol of a number, but of an operation, so Δ^2 does not denote a number multiplied by itself, but an operation repeated upon its own result; just as the logarithm of the logarithm of x might be written $\log^2 x$; $(\log x)^2$ being reserved to signify the square of the logarithm of x. We do not enlarge on this notation, as the subject is discussed in most treatises on

algebra.* Similarly the terms of the fourth column, or the differences of the second differences, have the prefix $\Delta\Delta\Delta$ abbreviated into Δ^3, so that $\Delta^2 y_1 - \Delta^2 y = \Delta^3 y$, etc.

When we have occasion to examine the results which arise from supposing Δx to diminish without limit, we use dx instead of Δx, dy instead of Δy, d^2y instead of $\Delta^2 y$, and so on. If we suppose this case, we can show that the ratio which the term in any column bears to its corresponding term in any preceding column, diminishes without limit. Take for example, d^2y and dy. The latter is $\varphi(x+dx) - \varphi x$, which, as we have often noticed already, is of the form $p\,dx + q\,(dx)^2 +$ etc., in which p, q, etc., are also functions of x. To obtain d^2y, we must, in this series, change x into $x + dx$, and subtract $p\,dx + q\,(dx)^2 +$ etc. from the result. But since p, q, etc., are functions of x, this change gives them the form

$$p + p'dx + \text{etc.}, \quad q + q'dx + \text{etc.};$$

so that d^2y is

$$(p + p'dx + \text{etc.})\,dx + (q + q'dx + \text{etc.})\,(dx)^2 + \text{etc.}$$
$$- (pdx + q\,(dx)^2 + \text{etc.})$$

in which the first power of dx is destroyed. Hence (pages 42–44), the ratio of d^2y to dx diminishes without limit, while that of d^2y to $(dx)^2$ has a finite limit, except in those particular cases in which the second power of dx is destroyed, in the previous subtraction, as well as the first. In the same way it may be shown that the ratio of d^3y to dx and $(dx)^2$ decreases without limit, while that of d^3y to $(dx)^3$ remains finite; and so

*The reference of the original text is to "the treatise on *Algebraical Expressions*," Number 105 of the Library of Useful Knowledge,—the same series in which the present work appeared. The first six pages of this treatise are particularly recommended by De Morgan in relation to the present point.—*Ed.*

on. Hence we have a succession of ratios $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$, etc., which tend towards finite limits when dx is diminished.

We now proceed to show that in the development of $\varphi(x+h)$, which has been shown to be of the form

$$\varphi x + \varphi' x\, h + \varphi'' x \frac{h^2}{2} + \varphi''' x \frac{h^3}{2.3} + \text{etc.},$$

in the same manner as $\varphi' x$ is the limit of $\frac{dy}{dx}$ (page 23), so $\varphi'' x$ is the limit of $\frac{d^2y}{dx^2}$, $\varphi''' x$ is that of $\frac{d^3y}{dy^3}$, and so forth.

From the manner in which the preceding table was formed, the following relations are seen immediately:

$$\begin{array}{ll} y_1 = y + \Delta y & \Delta y_1 = \Delta y + \Delta^2 y \\ y_2 = y_1 + \Delta y_1 & \Delta y_2 = \Delta y_1 + \Delta^2 y_1 \end{array}$$

$$\Delta^2 y_1 = \Delta^2 y + \Delta^3 y \text{ etc.}$$
$$\Delta^2 y_2 = \Delta^2 y_1 + \Delta^3 y_1 \text{ etc.}$$

Hence y_1, y_2, etc., can be expressed in terms of y, Δy, $\Delta^2 y$, etc. For $y_1 = y + \Delta y$; $y_2 = y_1 + \Delta y_1 = (y + \Delta y) + (\Delta y + \Delta^2 y) = y + 2\Delta y + \Delta^2 y$. In the same way $\Delta y_2 = \Delta y + 2\Delta^2 y + \Delta^3 y$; hence $y_3 = y_2 + \Delta y_2 = (y + 2\Delta y + \Delta^2 y) + (\Delta y + 2\Delta^2 y + \Delta^3 y) = y + 3\Delta y + 3\Delta^2 y + \Delta^3 y$. Proceeding in this way we have

$$\begin{aligned} y_1 &= y + \Delta y \\ y_2 &= y + 2\Delta y + \Delta^2 y \\ y_3 &= y + 3\Delta y + 3\Delta^2 y + \Delta^3 y \\ y_4 &= y + 4\Delta y + 6\Delta^2 y + 4\Delta^3 y + \Delta^4 y \\ y_5 &= y + 5\Delta y + 10\Delta^2 y + 10\Delta^3 y + 5\Delta^4 y + \Delta^5 y, \text{ etc.} \end{aligned}$$

from the whole of which it appears that y_n or $\varphi(x + n\Delta x)$ is a series consisting of y, Δy, etc., up to $\Delta^n y$, severally multiplied by the coefficients which occur in the expansion $(1 + a)^n$, or

$$y_n = \varphi(x + n\Delta x)$$
$$= y + n\Delta y + n\,\frac{n-1}{2}\,\Delta^2 y + n\,\frac{n-1}{2}\,\frac{n-2}{3}\,\Delta^3 y + \text{etc.}$$

Let us now suppose that x becomes $x+h$ by n equal steps; that is, x, $x+\frac{h}{n}$, $x+\frac{2h}{n}$, etc.... $x+\frac{nh}{n}$ or $x+h$, are the successive values of x, so that $n\Delta x = h$. Since the product of a number of factors is not altered by multiplying one of them, provided we divide another of them by the same quantity, multiply every factor which contains n by Δx, and divide the accompanying difference of y by Δx as often as there are factors which contain n, substituting h for $n\Delta x$, which gives

$$\varphi(x + n\Delta x) = y + n\Delta x\,\frac{\Delta y}{\Delta x} + n\Delta x\,\frac{n\Delta x - \Delta x}{2}\,\frac{\Delta^2 y}{(\Delta x)^2}$$
$$+ n\Delta x\,\frac{n\Delta x - \Delta x}{2}\,\frac{n\Delta x - 2\Delta x}{3}\,\frac{\Delta^3 y}{(\Delta x)^3} + \text{etc.}$$

or
$$\varphi(x+h) = y + h\,\frac{\Delta y}{\Delta x} + h\,\frac{h - \Delta x}{2}\,\frac{\Delta^2 y}{(\Delta x)^2}$$
$$+ h\,\frac{h - \Delta x}{2}\,\frac{h - 2\Delta x}{3}\,\frac{\Delta^3 y}{(\Delta x)^3} + \text{etc.}$$

If h remain the same, the more steps we make between x and $x+h$, the smaller will each of those steps be, and the number of steps may be increased, until each of them is as small as we please. We can therefore suppose Δx to decrease without limit, without affecting the truth of the series just deduced. Write dx for Δx, etc., and recollect that $h - dx$, $h - 2dx$, etc., continually approximate to h. The series then becomes

$$\varphi(x+h) = y + \frac{dy}{dx}\,h + \frac{d^2y}{dx^2}\,\frac{h^2}{2} + \frac{d^3y}{dx^3}\,\frac{h^3}{2.3} + \text{etc.}$$

in which, according to the view taken of the symbols $\frac{dy}{dx}$ etc. in pages 26–27, $\frac{dy}{dx}$ stands for the *limit* of the ratio of the increments, $\frac{dy}{dx}$ is $\varphi'x$, $\frac{d^2y}{dx^2}$ is $\varphi''x$, etc. According to the method proposed in pages 28–29, the series written above is the first term of the development of $\varphi(x+h)$, the remaining terms (which we might include under an additional $+$ etc.) being such as to diminish without limit in comparison with the first, when dx is diminished without limit. And we may show that the limit of $\frac{d^2y}{dx^2}$ is the differential coefficient of the limit of $\frac{dy}{dx}$; or if by these fractions themselves are understood their limits, that $\frac{d^2y}{dx^2}$ is the differential coefficient of $\frac{dy}{dx}$: for since dy, or $\varphi(x+dx) - \varphi x$, becomes $dy + d^2y$, when x is changed into $x + dx$; and since dx does not change in this process, $\frac{dy}{dx}$ will become $\frac{dy}{dx} + \frac{d^2y}{dx}$, or its increment is $\frac{d^2y}{dx}$. The ratio of this to dx is $\frac{d^2y}{(dx)^2}$, the limit of which, in the definition of page 22, is the differential coefficient of $\frac{dy}{dx}$. Similarly the limit of $\frac{d^3y}{dx^3}$ is the differential coefficient of the limit of $\frac{d^2y}{dx^2}$; and so on.

TOTAL AND PARTIAL DIFFERENTIAL COEFFICIENTS. IMPLICIT DIFFERENTIATION.

We now proceed to apply the principles laid down, to some cases in which the variable enters into its function in a less direct and more complicated manner.

For example, let z be a given function of x and y, and let y be another given function of x; so that z contains x both directly and indirectly; the latter as it contains y, which is a function of x. This will be the case if $z = x \log y$, where $y = \sin x$. If we were to substitute for y its value in terms of x, the value of z would then be a function of x only; in the instance just given it would be $x \log \sin x$. But if it be not convenient to combine the two equations at the beginning of the process, let us first consider z as a function of x and y, in which the two variables are independent. In this case, if x and y respectively receive the increments dx and dy, the whole increment of z, or $d.z$, (or at least that part which gives the limit of the ratios) is represented by

$$\frac{dz}{dx}dx + \frac{dz}{dy}dy.$$

If y be now considered as a function of x, the consequence is that dy, instead of being independent of dx, is a series of the form $pdx + q(dx)^2 +$ etc., in which p is the differential coefficient of y with respect to x. Hence

$$d.z = \frac{dz}{dx}dx + \frac{dz}{dy}pdx \text{ or } \frac{d.z}{dx} = \frac{dz}{dx} + \frac{dz}{dy}p,$$

in which the difference between $\frac{d.z}{dx}$ and $\frac{dz}{dx}$ is this, that in the second, x is only considered as varying where it is directly contained in z, or z is considered in the form in which it first appeared, as a function of x and y, where y is independent of x; in the first, or $\frac{d.z}{dx}$, the *total variation* of z is denoted, that is, y is now considered as a function of x, by which means if x become $x + dx$, z will receive a different increment

from that which it would have received, had y been independent of x. In the instance above cited, where $z = x \log y$ and $y = \sin x$, if the first equation be taken, and x becomes $x + dx$, y remaining the same, z becomes $x \log y + \log y\, dx$ or $\frac{dz}{dx}$ is $\log y$. If y only varies, since (page 20) z will then become

$$x \log y + x \frac{dy}{y} - \text{etc.},$$

$\frac{dz}{dy}$ is $\frac{x}{y}$. And $\frac{dy}{dx}$ is $\cos x$ when $y = \sin x$ (page 20). Hence $\frac{dz}{dx} + \frac{dz}{dy} p$, or $\frac{dz}{dx} + \frac{dz}{dy}\frac{dy}{dx}$ is $\log y + \frac{x}{y} \cos x$, or $\log \sin x + \frac{x}{\sin x} \cos x$. This is $\frac{d.z}{dx}$, which might have been obtained by a more complicated process, if $\sin x$ had been substituted for y, before the operation commenced. It is called the *complete* or *total* differential coefficient with respect to x, the word *total* indicating that every way in which z contains x has been used; in opposition to $\frac{dz}{dx}$, which is called the *partial* differential coefficient, x having been considered as varying only where it is directly contained in z.

Generally, the complete differential coefficient of z with respect to x, will contain as many terms as there are different ways in which z contains x. From looking at a complete differential coefficient, we may see in what manner the function contained its variable. Take, for example, the following,

$$\frac{d.z}{dx} = \frac{dz}{dx} + \frac{dz}{dy}\frac{dy}{dx} + \frac{dz}{da}\frac{da}{dy}\frac{dy}{dx} + \frac{dz}{da}\frac{da}{dx}.$$

Before proceeding to demonstrate this formula, we will collect from itself the hypothesis from which it

must have arisen. When x is contained in z, we shall say that z is a *direct** function of x. When x is contained in y, and y is contained in z, we shall say that z is an indirect function of x *through* y. It is evident that an indirect function may be reduced to one which is direct, by substituting for the quantities which contain x, their values in terms of x.

The first side of the equation $\frac{d.z}{dx}$ is shown by the point to be a complete differential coefficient, and indicates that z is a function of x in several ways; either directly, and indirectly through one quantity at least, or indirectly through several. If z be a direct function only, or indirectly through one quantity only, the symbol $\frac{dz}{dx}$, without the point, would represent its total differential coefficient with respect to x.

On the second side of the equation we see:

(1) $\frac{dz}{dx}$: which shows that z is a direct function of x, and is that part of the differential coefficient which we should get by changing x into $x + dx$ throughout z, not supposing any other quantity which enters into z to contain x.

(2) $\frac{dz}{dy}\frac{dy}{dx}$: which shows that z is an indirect function of x through y. If x and y had been supposed to vary independently of each other, the increment of z, (or those terms which give the limiting ratio of this increment to any other,) would have been $\frac{dz}{dx}dx + \frac{dz}{dy}dy$, in which, if dy had arisen from y being a func-

*It may be right to warn the student that this phraseology is new, to the best of our knowledge. The nomenclature of the Differential Calculus has by no means kept pace with its wants; indeed the same may be said of algebra generally. [Written in 1832.—*Ed.*]

tion of x, dy would have been a series of the form $pdx + q(dx)^2 +$ etc., of which only the differential co-efficient p would have appeared in the limit. Hence $\frac{dz}{dy}dy$ would have given $\frac{dz}{dy}p$, or $\frac{dz}{dy}\frac{dy}{dx}$.

(3) $\frac{dz}{da}\frac{da}{dy}\frac{dy}{dx}$: this arises from z containing a, which contains y, which contains x. If z had been differentiated with respect to a only, the increment would have been represented by $\frac{dz}{da}da$; if da had arisen from an increment of y, this would have been expressed by $\frac{dz}{da}\frac{da}{dy}dy$; if y had arisen from an increment given to x, this would have been expressed by $\frac{dz}{da}\frac{da}{dy}\frac{dy}{dx}dx$, which, after dx has been struck out, is the part of the differential coefficient answering to that increment.

(4) $\frac{dz}{da}\frac{da}{dx}$: arising from a containing x directly, and z therefore containing x indirectly through a.

Hence z is directly a function of x, y, and a, of which y is a function of x, and a of y and x.

If we suppose x, y and a to vary independently, we have

$$d.z = \frac{dz}{dx}dx + \frac{dz}{dy}dy + \frac{dz}{da}da + \text{etc.} \quad \text{(pages 28–29)}.$$

But as a varies as a function of y and x,

$$da = \frac{da}{dx}dx + \frac{da}{dy}dy.$$

If we substitute this instead of da, and divide by dx, taking the limit of the ratios, we have the result first given.

For example, let (1) $z = x^2ya^3$, (2) $y = x^2$, and (3) $a = x^3y$. Taking the first equation only, and substi-

tuting $x + dx$ for x etc., we find $\frac{dz}{dx} = 2xya^8$, $\frac{dz}{dy} = x^2a^3$, and $\frac{dz}{da} = 3x^2ya^2$. From the second $\frac{dy}{dx} = 2x$, and from the third $\frac{da}{dx} = 3x^2y$, and $\frac{da}{dy} = x^3$. Substituting these in the value of $\frac{d.z}{dx}$, we find

$$\begin{aligned}\frac{d.z}{dx} \text{ or } \frac{dz}{dx} &+ \frac{dz}{dy}\frac{dy}{dx} + \frac{dz}{da}\frac{da}{dy}\frac{dy}{dx} + \frac{dz}{da}\frac{da}{dx} \\ &= 2xya^8 + x^2a^3 \times 2x + 3x^2ya^2 \times x^3 \times 2x + 3x^2ya^2 \times 3x^2y \\ &= 2xya^3 + 2x^3a^3 + 6x^6ya^2 + 9x^4y^2a^2\end{aligned}$$

If for y and a in the first equation we substitute their values x^2 and x^3y, or x^5, we have $z = x^{19}$, the differential coefficient of which $19x^{18}$. This is the same as arises from the formula just obtained, after x^2 and x^5 have been substituted for y and a; for this formula then becomes

$$2x^{18} + 2x^{18} + 6x^{18} + 9x^{18} \text{ or } 19x^{18}.$$

In saying that z is a function of x and y, and that y is a function of x, we have first supposed x to vary, y remaining the same. The student must not imagine that y is *then* a function of x; for if so, it would vary when x varied. There are two parts of the total differential coefficient, arising from the direct and indirect manner in which z contains x. That these two parts may be obtained separately, and that their sum constitutes the complete differential coefficient, is the theorem we have proved. The first part $\frac{dz}{dx}$ is what *would* have been obtained if y had *not* been a function of x; and on this supposition we therefore proceed to find it. The other part $\frac{dz}{dy}\frac{dy}{dx}$ is the product (1) of $\frac{dz}{dy}$, which would have resulted from a variation of y only, not considered as a function of x; and (2) of

$\frac{dy}{dx}$, the coefficient which arises from considering y as a function of x. These partial suppositions, however useful in obtaining the total differential coefficient, cannot be separately admitted or used, except for this purpose; since if y be a function of x, x and y must vary together.

If z be a function of x in various ways, the theorem obtained may be stated as follows:

Find the differential coefficient belonging to each of the ways in which z will contain x, as if it were the only way; the sum of these results (with their proper signs) will be the total differential coefficient.

Thus, if z only contains x indirectly through y, $\frac{dz}{dx}$ is $\frac{dz}{dy}\frac{dy}{dx}$. If z contains a, which contains b, which contains x, $\frac{dz}{dx}=\frac{dz}{da}\frac{da}{db}\frac{db}{dx}$.

This theorem is useful in the differentiation of complicated functions; for example, let $z=\log(x^2+a^2)$. If we make $y=x^2+a^2$, we have $z=\log y$, and $\frac{dz}{dy}=\frac{1}{y}$; while from the first equation $\frac{dy}{dx}=2x$. Hence $\frac{dz}{dx}$ or $\frac{dz}{dy}\frac{dy}{dx}$ is $\frac{2x}{y}$ or $\frac{2x}{x^2+a^2}$.

If $z=\log\log\sin x$, or the logarithm of the logarithm of $\sin x$, let $\sin x=y$ and $\log y=a$; whence $z=\log a$, and contains x, because a contains y, which contains x. Hence

$$\frac{dz}{dx}=\frac{dz}{da}\frac{da}{dy}\frac{dy}{dx};$$

but since $z=\log a$,

$$\frac{dz}{da}=\frac{1}{a};$$

since $a = \log y$,

$$\frac{da}{dy} = \frac{1}{y};$$

and since $y = \sin x$,

$$\frac{dy}{dx} = \cos x.$$

Hence

$$\frac{dz}{dx} = \frac{dz}{da}\frac{da}{dy}\frac{dy}{dx} = \frac{1}{a}\frac{1}{y}\cos x = \frac{\cos x}{\log \sin x \sin x}.$$

We now put some rules in the form of applications of this theorem, though they may be deduced more simply.

APPLICATIONS OF THE PRECEDING THEOREM.

(1) Let $z = ab$, where a and b are functions of x. The general formula, since z contains x indirectly through a and b, is (in this case as well as in those which follow,)

$$\frac{dz}{dx} = \frac{dz}{da}\cdot\frac{da}{dx} + \frac{dz}{db}\frac{db}{dx}.$$

We must leave $\frac{da}{dx}$ and $\frac{db}{dx}$ as we find them, until we know *what* functions a and b are of x; but as we know what function z is of a and b, we substitute for $\frac{dz}{da}$ and $\frac{dz}{db}$. Since $z = ab$, if a becomes $a + da$, z becomes $ab + b\,da$, whence $\frac{dz}{da} = b$. In this case, and part of the following, the limiting ratio of the increments is the same as that of the increments themselves. Similarly $\frac{dz}{db} = a$, whence from $z = ab$ follows

$$\frac{dz}{dx} = b\frac{da}{dx} + a\frac{db}{dx}.$$

(2) Let $z = \frac{a}{b}$. If a become $a + da$, z becomes $\frac{a + da}{b}$ or $\frac{a}{b} + \frac{da}{b}$, and $\frac{dz}{da}$ is $\frac{1}{b}$. If b become $b + db$, z becomes $\frac{a}{b + db}$ or $\frac{a}{b} - \frac{adb}{b^2} +$ etc., whence $\frac{dz}{db}$ is $-\frac{a}{b^2}$. Hence from $z = \frac{a}{b}$ follows

$$\frac{dz}{dx} = \frac{1}{b}\frac{da}{dx} - \frac{a}{b^2}\frac{db}{dx} = \frac{b\frac{da}{dx} - a\frac{db}{dx}}{b^2}.$$

(3) Let $z = a^b$. Here $(a + da)^b = a^b + ba^{b-1}da$ + etc. (page 21), whence $\frac{dz}{da} = ba^{b-1}$. Again, $a^{b+db} = a^b a^{db} = a^b(1 + \log a\, db +$ etc.) whence $\frac{dz}{db} = a^b \log a$. Therefore from $z = a^b$ follows

$$\frac{dz}{dx} = ba^{b-1}\frac{da}{dx} + a^b \log a\, \frac{db}{dx}.$$

INVERSE FUNCTIONS.

If y be a function of x, such as $y = \varphi x$, we may, by solution of the equation, determine x in terms of y, or produce another equation of the form $x = \psi y$. For example, when $y = x^2$, $x = y^{\frac{1}{2}}$. It is not necessary that we should be able to solve the equation $y = \varphi x$ in finite terms, that is, so as to give a value of x without infinite series; it is sufficient that x can be so expressed that the value of x corresponding to any value of y may be found as near as we please from $x = \psi y$, in the same manner as the value of y corresponding to any value of x is found from $y = \varphi x$.

The equations $y = \varphi x$, and $x = \psi y$, are connected, being, in fact, the same relation in different forms; and if the value of y from the first be substituted in

the second, the second becomes $x = \psi(\varphi x)$, or as it is more commonly written, $\psi\varphi x$. That is, the effect of the operation or set of operations denoted by ψ is destroyed by the effect of those denoted by φ; as in the instances $(x^2)^{\frac{1}{2}}$, $(x^3)^{\frac{1}{3}}$, $\varepsilon^{\log x}$, angle whose sine is $(\sin x)$, etc., each of which is equal to x.

By differentiating the first equation $y = \varphi x$, we obtain $\frac{dy}{dx} = \varphi' x$, and from the second $\frac{dx}{dy} = \psi' y$. But whatever values of x and y together satisfy the first equation, satisfy the second also; hence, if when x becomes $x + dx$ in the first, y becomes $y + dy$; the same $y + dy$ substituted for y in the second, will give the same $x + dx$. Hence $\frac{dx}{dy}$ as deduced from the second, and $\frac{dy}{dx}$ as deduced from the first, are reciprocals for every value of dx. The limit of one is therefore the reciprocal of the limit of the other; the student may easily prove that if a is always equal to $\frac{1}{b}$, and if a continually approaches to the limit α, while b at the same time approaches the limit β, α is equal to $\frac{1}{\beta}$. But $\frac{dx}{dy}$ or $\psi' y$, deduced from $x = \psi y$, is expressed in terms of y, while $\frac{dy}{dx}$ or $\varphi' x$, deduced from $y = \varphi x$ is expressed in terms of x. Therefore $\psi' y$ and $\varphi' x$ are reciprocals for all such values of x and y as satisfy either of the two first equations.

For example let $y = \varepsilon^x$, from which $x = \log y$. From the first (page 20) $\frac{dy}{dx} = \varepsilon^x$; from the second $\frac{dx}{dy} = \frac{1}{y}$; and it is evident that ε^x and $\frac{1}{y}$ are reciprocals, whenever $y = \varepsilon^x$.

If we differentiate the above equations twice, we get

$\frac{d^2y}{dx^2} = \varphi''x$, and $\frac{d^2x}{dy^2} = \psi''x$. There is no very obvious analogy between $\frac{d^2y}{dx^2}$ and $\frac{d^2x}{dy^2}$; indeed no such appears from the method in which these coefficients were first formed. Turn to the table in page 90, and substitute d for Δ throughout, to indicate that the increments may be taken as small as we please. We there substitute in φx what we will call a set of *equidistant* values of x, or values in arithmetical progression, viz., x, $x + dx$, $x + 2dx$, etc. The resulting values of y, or y, y_1, etc., are not equidistant, except in one function only, when $y = ax + b$, where a and b are constant. Therefore dy, dy_1, etc., are not equal; whence arises the next column of second differences, or d^2y, d^2y_1, etc. The limiting ratio of d^2y to $(dx)^2$, expressed by $\frac{d^2y}{dx^2}$, is the second differential coefficient of y with respect to x. If from $y = \varphi x$ we deduce $x = \psi y$, and take a set of equidistant values of y, viz., y, $y + dy$, $y + 2dy$, etc., to which the corresponding values of x are x, x_1, x_2, etc., a similar table may be formed, which will give dx, dx_1, etc., d^2x, d^2x_1, etc., and the limit of the ratio of d^2x to $(dy)^2$ or $\frac{d^2x}{dy^2}$ is the second differential coefficient of x with respect to y. These are entirely different suppositions, dx being given in the first table, and dy varying; while in the second dy is given and dx varies. We may show how to deduce one from the other as follows:

When, as before, $y = \varphi x$ and $x = \psi y$, we have

$$\frac{dy}{dx} = \varphi' x = \frac{1}{\psi' y} = \frac{1}{p},$$

if $\psi' y$ be called p. Calling this u, and considering it

as a function of x from containing p, which contains y, which contains x, we have

$$\frac{du}{dp}\,\frac{dp}{dy}\,\frac{dy}{dx}$$

for its differential coefficient with respect to x. But since

$$u=\frac{1}{p},$$

therefore

$$\frac{du}{dp}=-\frac{1}{p^2};$$

since $p=\psi'y$, therefore

$$\frac{dp}{dy}=\psi''y;$$

and $\psi''y$ is the differential coefficient of $\psi'y$, and is $\frac{d^2x}{dy^2}$. Also $\frac{1}{p^2}$ is

$$\frac{1}{(\psi'y)^2} \text{ or } (\varphi'x)^2 \text{ or } \left(\frac{dy}{dx}\right)^2.$$

Hence the differential coefficient of u or $\frac{dy}{dx}$, with respect to x, which is $\frac{d^2y}{dx^2}$, is also

$$-\left(\frac{dy}{dx}\right)^2\frac{d^2x}{dy^2}\,\frac{dy}{dx} \text{ or } -\left(\frac{dy}{dx}\right)^3\frac{d^2x}{dy^2}.$$

If $y=\varepsilon^x$, whence $x=\log y$, we have $\frac{dy}{dx}=\varepsilon^x$ and $\frac{d^2y}{dx^2}=\varepsilon^x$. But $\frac{dx}{dy}=\frac{1}{y}$ and $\frac{d^2x}{dy^2}=-\frac{1}{y^2}$. Therefore

$$-\left(\frac{dy}{dx}\right)^3\frac{d^2x}{dy^2} \text{ is } -\varepsilon^{3x}\left(-\frac{1}{y^2}\right) \text{ or } \frac{\varepsilon^{3x}}{y^2} \text{ or } \frac{\varepsilon^{3x}}{\varepsilon^{2x}},$$

which is ε^x, the value just found for $\frac{d^2y}{dx^2}$.

In the same way $\frac{d^3y}{dx^3}$ might be expressed in terms of $\frac{dx}{dy}$, $\frac{d^2x}{dy^2}$, and $\frac{d^3x}{dy^3}$; and so on.

IMPLICIT FUNCTIONS.

The variable which appears in the denominator of the differential coefficients is called the *independent* variable. In any function, one quantity at least is changed at pleasure; and the changes of the rest, with the limiting ratio of the changes, follow from the form of the function. The number of independent variables depends upon the number of quantities which enter into the equations, and upon the number of equations which connect them. If there be only one equation, all the variables except one are independent, or may be changed at pleasure, without ceasing to satisfy the equation; for in such a case the common rules of algebra tell us, that as long as one quantity is left to be determined from the rest, it can be determined by one equation; that is, the values of all but one are at our pleasure, it being still in our power to satisfy one equation, by giving a proper value to the remaining one. Similarly, if there be two equations, all variables except two are independent, and so on. If there be two equations with two unknown quantities only, there are no variables; for by algebra, a finite number of values, and a finite number only, can satisfy these equations; whereas it is the nature of a variable to receive any value, or at least any value which will not give impossible values for other variables. If then there be m equations containing n variables, (n must be greater than m), we have $n-m$ independent variables, to each of which

we may give what values we please, and by the equations, deduce the values of the rest. We have thus various sets of differential coefficients, arising out of the various choices which we may make of independent variables.

If, for example, a, b, x, y, and z, being variables, we have

$$\varphi(a, b, x, y, z,) = 0,$$
$$\psi(a, b, x, y, z,) = 0,$$
$$\chi(a, b, x, y, z,) = 0,$$

we have two independent variables, which may be either x and y, x and z, a and b, or any other combination. If we choose x and y, we should determine a, b, and z in terms of x and y from the three equations; in which case we can obtain

$$\frac{da}{dx}, \frac{da}{dy}, \frac{db}{dx}, \text{ etc.}$$

When y is a function of x, as in $y = \varphi x$, it is called an *explicit* function of x. This equation tells us not only that y is a function of x, but also what function it is. The value of x being given, nothing more is necessary to determine the corresponding value of y, than the substitution of the value of x in the several terms of φx.

But it may happen that though y is a function of x, the relation between them is contained in a form from which y must be deduced by the solution of an equation. For example, in $x^2 - xy + y^2 = a$, when x is known, y must be determined by the solution of an equation of the second degree. Here, though we know that y must be a function of x, we do not know, without further investigation, what function it is. In this case y is said to be *implicitly* a function of x, or an im-

plicit function. By bringing all the terms on one side of the equation, we may always reduce it to the form $\varphi(x, y) = 0$. Thus, in the case just cited, we have $x^2 - xy + y^2 - a = 0$.

We now want to deduce the differential coefficient $\frac{dy}{dx}$ from an equation of the form $\varphi(x, y) = 0$. If we take the equation $u = \varphi(x, y)$, in which when x and y become $x + dx$ and $y + dy$, u becomes $u + du$, we have, by our former principles,

$$du = u'dx + u_{,}dy + \text{etc., (page 82)},$$

in which u' and $u_{,}$ can be directly obtained from the equation, as in page 82. Here x and y are independent, as also dx and dy; whatever values are given to them, it is sufficient that u and du satisfy the two last equations. But if x and y must be always so taken that u may $= 0$, (which is implied in the equation $\varphi(x, y) = 0$,) we have $u = 0$, and $du = 0$; and this, whatever may be the values of dx and dy. Hence dx and dy are connected by the equation

$$0 = u'dx + u_{,}dy + \text{etc.},$$

and their limiting ratio must be obtained by the equation

$$u'dx + u_{,}dy = 0, \quad \text{or } \frac{dy}{dx} = -\frac{u'}{u_{,}};$$

y and x are no longer independent; for, one of them being given, the other must be so taken that the equation $\varphi(x, y) = 0$ may be satisfied. The quantities u' and $u_{,}$ we have denoted by $\frac{du}{dx}$ and $\frac{du}{dy}$, so that

$$\frac{dy}{dx} = -\frac{\frac{du}{dx}}{\frac{du}{dy}} \tag{1}$$

We must again call attention to the different meanings of the same symbol du in the numerator and denominator of the last fraction. Had du, dx, and dy been common algebraical quantities, the first meaning the same thing throughout, the last equation would not have been true until the negative sign had been removed. We will give an instance in which du shall mean the same thing in both.

Let $u = \varphi(x)$, and let $u = \psi y$, in which two equations is implied a third $\varphi x = \psi y$; and y is a function of x. Here, x being given, u is known from the first equation; and u being known, y is known from the second. Again, x and dx being given, du, which is $\varphi(x + dx) - \varphi x$ is known, and being substituted in the result of the second equation, we have $du = \psi(y + dy) - \psi y$, which dy must be so taken as to satisfy. From the first equation we deduce $du = \varphi' x\, dx +$ etc. and from the second $du = \psi' y\, dy +$ etc., whence

$$\varphi' x\, dx + \text{etc.} = \psi' y\, dy + \text{etc.};$$

the *etc.* only containing terms which disappear in finding the limiting ratios. Hence,

$$\frac{dy}{dx} = \frac{\varphi' x}{\psi' y} = \frac{\frac{du}{dx}}{\frac{du}{dy}} \tag{2}$$

a result in accordance with common algebra.

But the equation (1) was obtained from $u = \varphi(x, y)$, on the supposition that x and y were always so taken that u should $= 0$, while (2) was obtained from $u = \varphi(x)$ and $u = Sy$, in which no new supposition can be made; since one more equation between u, x, and y would give three equations connecting these three quantities, in which case they would cease to be variable (page 106).

As an example of (1) let $xy-x=1$, or $xy-x-1=0$. From $u=xy-x-1$ we deduce (page 81) $\frac{du}{dx}=y-1$, $\frac{du}{dy}=x$; whence, by equation (1),

$$\frac{dy}{dx}=-\frac{y-1}{x} \qquad (3).$$

By solution of $xy-x=1$, we find $y=1+\frac{1}{x}$, and

$$dy=\left(1+\frac{1}{x+dx}\right)-\left(1+\frac{1}{x}\right)=-\frac{dx}{x^2}+\text{etc.}^*$$

Hence $\frac{dy}{dx}$ (meaning the limit) is $-\frac{1}{x^2}$, which will also be the result of (3) if $1+\frac{1}{x}$ be substituted for y.

FLUXIONS, AND THE IDEA OF TIME.

To follow this subject farther would lead us beyond our limits; we will therefore proceed to some observations on the differential coefficient, which, at this stage of his progress, may be of use to the student, who should never take it for granted that because he has made some progress in a science, he understands the first principles, which are often, if not always, the last to be learned well. If the mind were so constituted as to receive with facility any perfectly new idea, as soon as the same was legitimately applied in mathematical demonstration, it would doubtless be an advantage not to have any notion upon a mathematical subject, previous to the time when it is to become a subject of consideration after a strictly mathematical method.

This not being the case, it is a cause of embarrassment to the student, that he is introduced at once to a definition so refined as that of the limiting ratio which

* See page 26.

the increment of a function bears to the increment of its variable. Of this he has not had that previous experience, which is the case in regard to the words *force*, *velocity*, or *length*. Nevertheless, he can easily conceive a mathematical quantity in a state of continuous increase or decrease, such as the distance between two points, one of which is in motion. The number which represents this line (reference being made to a given linear unit) is in a corresponding state of increase or decrease, and so is every function of this number, or every algebraical expression in the formation of which it is required. And the nature of the change which takes place in the function, that is, whether the function will increase or decrease when the variable increases; whether that increase or decrease corresponding to a given change in the variable will be smaller or greater, etc., depends on the manner in which the variable enters as a component part of its function.

Here we want a new word, which has not been invented for the world at large, since none but mathematicians consider the subject; which word, if the change considered were change of place, depending upon change of time, would be *velocity*. Newton adopted this word, and the corresponding idea, expressing many numbers in succession, instead of at once, by supposing a point to generate a straight line by its motion, which line would at different instants contain any different numbers of linear units.

To this it was objected that the idea of *time* is introduced, which is foreign to the subject. We may answer that the notion of time is only necessary, inasmuch as we are not able to consider more than one thing at a time. Imagine the diameter of a circle di-

vided into a million of equal parts, from each of which a perpendicular is drawn meeting the circle. A mind which could at a view take in every one of these lines, and compare the differences between every two contiguous perpendiculars with one another, could, by subdividing the diameter still further, prove those propositions which arise from supposing a point to move uniformly along the diameter, carrying with it a perpendicular which lengthens or shortens itself so as always to have one extremity on the circle. But we, who cannot consider all these perpendiculars at once, are obliged to take one after another. If one perpendicular only were considered, and the differential coefficient of that perpendicular deduced, we might certainly appear to avoid the idea of time; but if all the states of a function are to be considered, corresponding to the different states of its variable, we have no alternative, with our bounded faculties, but to consider them in succession; and succession, disguise it as we may, is the identical idea of time introduced in Newton's Method of Fluxions.

THE DIFFERENTIAL COEFFICIENT CONSIDERED WITH RESPECT TO ITS MAGNITUDE.

The differential coefficient corresponding to a particular value of the variable, is, if we may use the phrase, the *index* of the change which the function would receive if the value of the variable were increased. Every value of the variable, gives not only a different value to the function, but a different quantity of increase or decrease in passing to what we may call *contiguous* values, obtained by a given increase of the variable.

If, for example, we take the common logarithm of

x, and let x be 100, we have common log $100 = 2$. If x be increased by 2, this gives common log $102 = 2\cdot 0086002$, the ratio of the increment of the function to that of the variable being that of $\cdot 0086002$ to 2, or $\cdot 0043001$. In passing from 1000 to 1003, we have the logarithms 3 and $3\cdot 0013009$, the above-mentioned ratio being $\cdot 0004336$, little more than a tenth of the former. We do not take the increments themselves, but the proportion they bear to the changes in the variable which gave rise to them; so in estimating the rate of motion of two points, we either consider lengths described in the same time, or if that cannot be done, we judge, not by the lengths described in different times, but by the proportion of those lengths to the times, or the proportions of the units which express them.

The above rough process, though from it some might draw the conclusion that the logarithm of x is increasing faster when $x = 100$ than when $x = 1000$, is defective; for, in passing from 100 to 102, the change of the logarithm is not a sufficient index of the change which is taking place when x is 100; since, for any thing we can be supposed to know to the contrary, the logarithm might be decreasing when $x = 100$, and might afterwards begin to increase between $x = 100$ and $x = 102$, so as, on the whole, to cause the increase above mentioned. The same objection would remain good, however small the increment might be, which we suppose x to have. If, for example, we suppose x to change from $x = 100$ to $x = 100\cdot 00001$, which increases the logarithm from 2 to $2\cdot 00000004343$, we cannot yet say but that the logarithm may be decreasing when $x = 100$, and may begin to increase between $x = 100$ and $x = 100\cdot 00001$.

In the same way, if a point is moving, so that at the end of 1 second it is at 3 feet from a fixed point, and at the end of 2 seconds it is at 5 feet from the fixed point, we cannot say which way it is moving at the end of one second. *On the whole*, it increases its distance from the fixed point in the second second; but it is possible that at the end of the first second it may be moving back towards the fixed point, and may turn the contrary way during the second second. And the same argument holds, if we attempt to ascertain the way in which the point is moving by supposing any finite portion to elapse after the first second. But if on adding any interval, *however small*, to the first second, the moving point does, during that interval, increase its distance from the fixed point, we can then certainly say that at the end of the first second the point is moving from the fixed point.

On the same principle, we cannot say whether the logarithm of x is increasing or decreasing when x increases and becomes 100, unless we can be sure that any increment, however small, added to x, will increase the logarithm. Neither does the ratio of the increment of the function to the increment of its variable furnish any distinct idea of the change which is taking place when the variable has attained or is passing through a given value. For example, when x passes from 100 to 102, the difference between $\log 102$ and $\log 100$ is the united effect of all the changes which have taken place between $x=100$ and $x=100\frac{1}{10}$; $x=100\frac{1}{10}$ and $x=100\frac{2}{10}$, and so on. Again, the change which takes place between $x=100$ and $x=100\frac{1}{10}$ may be further compounded of those which take place between $x=100$ and $x=100\frac{1}{100}$; $x=100\frac{1}{100}$ and $x=100\frac{2}{100}$, and so on. The objection

becomes of less force as the increment diminishes, but always exists unless we take the limit of the ratio of the increments, instead of that ratio.

How well this answers to our previously formed ideas on such subjects as direction, velocity, and force, has already appeared.

THE INTEGRAL CALCULUS.

We now proceed to the Integral Calculus, which is the inverse of the Differential Calculus, as will afterwards appear.

We have already shown, that when two functions *increase* or *decrease* without limit, their *ratio* may either increase or decrease without limit, or may tend to some finite limit. Which of these will be the case depends upon the manner in which the functions are related to their variable and to one another.

This same proposition may be put in another form, as follows: If there be two functions, the first of which *decreases* without limit, on the same supposition which makes the second *increase* without limit, the *product* of the two may either remain finite, and never exceed a certain finite limit; or it may increase without limit, or diminish without limit.

For example, take $\cos\theta$ and $\tan\theta$. As the angle θ *approaches* a right angle, $\cos\theta$ diminishes without limit; it is nothing when θ *is* a right angle; and any fraction being named, θ can be taken so near to a right angle that $\cos\theta$ shall be smaller. Again, as θ approaches to a right angle, $\tan\theta$ increases without limit; it is called *infinite* when θ is a right angle, by which we mean that, let any number be named, however great, θ can be taken so near a right angle that $\tan\theta$ shall be greater. Nevertheless the product $\cos\theta\times$

$\tan\theta$, of which the first factor diminishes without limit, while the second increases without limit, is always finite, and tends towards the limit 1; for $\cos\theta \times \tan\theta$ is always $\sin\theta$, which last approaches to 1 as θ approaches to a right angle, and *is* 1 when θ *is* a right angle.

Generally, if A diminishes without limit at the same time as B increases without limit, the product AB may, and often will, tend towards a finite limit. This product AB is the representative of A divided by $\frac{1}{B}$ or the ratio of A to $\frac{1}{B}$. If B increases without limit, $\frac{1}{B}$ decreases without limit; and as A also decreases without limit, the ratio of A to $\frac{1}{B}$ may have a finite limit. But it may also diminish without limit; as in the instance of $\cos^2\theta \times \tan\theta$, when θ approaches to a right angle. Here $\cos^2\theta$ diminishes without limit, and $\tan\theta$ increases without limit; but $\cos^2\theta \times \tan\theta$ being $\cos\theta \times \sin\theta$, or a diminishing magnitude multiplied by one which remains finite, diminishes without limit. Or it may increase without limit, as in the case of $\cos\theta \times \tan^2\theta$, which is also $\sin\theta \times \tan\theta$; which last has one factor finite, and the other increasing without limit. We shall soon see an instance of this.

If we take any numbers, such as 1 and 2, it is evident that between the two we may interpose any number of fractions, however great, either in arithmetical progression, or according to any other law. Suppose, for example, we wish to interpose 9 fractions in arithmetical progression between 1 and 2. These are $1\frac{1}{10}$, $1\frac{2}{10}$, etc., up to $1\frac{9}{10}$; and, generally, if m fractions in arithmetical progression be interposed between a and $a+h$, the complete series is

$$a,\ a+\frac{h}{m+1},\ a+\frac{2h}{m+1},\ \text{etc.}\ \ldots\ldots\ldots\ldots\ \text{up to}\ a+\frac{mh}{m+1},\ a+h \quad (1)$$

The sum of these can evidently be made as great as we please, since no one is less than the given quantity a, and the number is as great as we please. Again, if we take φx, any function of x, and let the values just written be successively substituted for x, we shall have the series

$$\varphi a,\ \varphi\left(a+\frac{h}{m+1}\right),\ \varphi\left(a+\frac{2h}{m+1}\right),\ \text{etc.,}\ \ldots\ldots\ldots\ldots\ \text{up to}\ \varphi(a+h) \quad (2);$$

the sum of which may, in many cases, also be made as great as we please by sufficiently increasing the number of fractions interposed, that is, by sufficiently increasing m. But though the two sums increase without limit when m increases without limit, it does not therefore follow that their ratio increases without limit; indeed we can show that this cannot be the case when all the separate terms of (2) remain finite.

For let A be greater than any term in (2), whence, as there are $(m+2)$ terms, $(m+2)\mathrm{A}$ is greater than their sum. Again, every term of (1), except the first, being greater than a, and the terms being $m+2$ in number, $(m+2)a$ is less than the sum of the terms in (1). Consequently,

$$\frac{(m+2)\mathrm{A}}{(m+2)a} \text{ is greater than the ratio } \frac{\text{sum of terms in (2)}}{\text{sum of terms in (1)}},$$

since its numerator is greater than the last numerator, and its denominator less than the last denominator. But

$$\frac{(m+2)A}{(m+2)a} = \frac{A}{a},$$

which is independent of m, and is a finite quantity. Hence the ratio of the sums of the terms is always finite, whatever may be the number of terms, at least unless the terms in (2) increase without limit.

As the number of interposed values increases, the interval or difference between them diminishes; if, therefore, we multiply this difference by the sum of the values, or form

$$\frac{h}{m+1}\left[\varphi a + \varphi\left(a + \frac{h}{m+1}\right) + \varphi\left(a + \frac{2h}{m+1}\right) \ldots\ldots + \varphi(a+h)\right]$$

we have a product, one term of which diminishes, and the other increases, when m is increased. The product *may* therefore remain finite, or never pass a certain limit, when m is increased without limit, and we shall show that this *is* the case.

As an example, let the given function of x be x^2, and let the intermediate values of x be interposed between $x=a$ and $x=a+h$. Let $v = \frac{h}{m+1}$, whence the above-mentioned product is

$$v\left\{a^2 + (a+v)^2 + (a+2v)^2 + \ldots\ldots\ldots\ldots \ldots\ldots\ldots\ldots + \left(a+(m+1)v\right)^2\right\} =$$

$$(m+2)va^2 + 2av^2\{1+2+3+\ldots+(m+1)\} + v^3\{1^2+2^2+3^2+\ldots+(m+1)^2\};$$

of which, $1+2+\ldots\ldots+(m+1) = \frac{1}{2}(m+1)(m+2)$ and (page 73), $1^2+2^2+\ldots\ldots+(m+1)^2$ approaches without limit to a ratio of equality with $\frac{1}{3}(m+1)^3$, when m is increased without limit. Hence this last sum may be put under the form $\frac{1}{3}(m+1)^3(1+\alpha)$,

where α diminishes without limit when m is increased without limit. Making these substitutions, and putting for v its value $\frac{h}{m+1}$, the above expression becomes

$$\frac{m+2}{m+1}ha^2 + \frac{m+2}{m+1}h^2a + (1+\alpha)\frac{h^3}{3},$$

in which $\frac{m+2}{m+1}$ has the limit 1 when m increases without limit, and $1+\alpha$ has also the limit 1, since, in that case, α diminishes without limit. Therefore the limit of the last expression is

$$ha^2 + h^2a + \frac{h^3}{3} \text{ or } \frac{(a+h)^3 - a^3}{3}.$$

This result may be stated as follows: If the variable x, setting out from a value a, becomes successively $a + dx$, $a + 2dx$, etc., until the total increment is h, the smaller dx is taken, the more nearly will the sum of all the values of x^2dx, or $a^2dx + (a+dx)^2dx + (a+2dx)^2dx +$ etc., be equal to

$$\frac{(a+h)^3 - a^3}{3},$$

and to this the aforesaid sum may be brought within any given degree of nearness, by taking dx sufficiently small.

This result is called the *integral* of x^2dx, between the limits a and $a+h$, and is written $\int x^2dx$, when it is not necessary to specify the limits, and $\int_a^{a+h} x^2dx$, or* $\int x^2dx_a^{a+h}$, or $\int x^2dx_{x=a}^{x=a+h}$ in the contrary case. We

*This notation $\int x^2dx_a^{a+h}$ appears to me to avoid the objections which may be raised against $\int_{a+h}^a x^2dx$ as contrary to analogy, which would require that $\int^2 x^2dx^2$ should stand for the second integral of x^2dx. It will be found convenient in such integrals as $\int zdx_b^a\, dy_0^{\phi x}$. There is as yet no general agreement on this point of notation.—*De Morgan*, 1832.

now proceed to show the connexion of this process with the principles of the Differential Calculus.

CONNEXION OF THE INTEGRAL WITH THE DIFFERENTIAL CALCULUS.

Let x have the successive values a, $a + dx$, $a + 2dx$, etc., ... up to $a + mdx$, or $a + h$, h being a given quantity, and dx the m^{th} part of h, so that as m is increased without limit, dx is diminished without limit. Develop the successive values φx, or φa, $\varphi(a + dx)$.... (page 21),

$$\begin{array}{llllll}
\phi a & = \phi a & & & & \\
\phi(a+dx) & = \phi a + \phi' a\, dx & + \phi'' a \dfrac{(dx)^2}{2} & + \phi''' a \dfrac{(dx)^3}{2.3} & + \text{etc.} \\
\phi(a+2dx) & = \phi a + \phi' a\, 2dx & + \phi'' a \dfrac{(2dx)^2}{2} & + \phi''' a \dfrac{(2dx)^3}{2.3} & + \text{etc.} \\
\phi(a+3dx) & = \phi a + \phi' a\, 3dx & + \phi'' a \dfrac{(3dx)^2}{2} & + \phi''' a \dfrac{(3dx)^3}{2.3} & + \text{etc.} \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\phi(a+mdx) & = \phi a + \phi' a\, mdx & + \phi'' a \dfrac{(mdx)^2}{2} & + \phi''' a \dfrac{(mdx)^3}{2.3} & + \text{etc.}
\end{array}$$

If we multiply each development by dx and add the results, we have a series made up of the following terms, arising from the different columns,

$$\begin{array}{lll}
\varphi a & \times & mdx \\
\varphi' a & \times (1 + 2 + 3 + \ldots + m) & (dx)^2 \\
\varphi'' a & \times (1^2 + 2^2 + 3^2 + \ldots\ldots m^2) & \dfrac{(dx)^3}{2} \\
\varphi''' a & \times (1^3 + 2^3 + 3^3 + \ldots + m^3) & \dfrac{(dx)^4}{2.3} \quad \text{etc.}
\end{array}$$

and, as in the last example, we may represent (page 73),

$$\begin{array}{ll}
1 + 2 + 3 + \ldots\ldots + m & \text{by } \tfrac{1}{2} m^2 (1 + \alpha) \\
1^2 + 2^2 + 3^2 + \ldots\ldots + m^2 & \ldots \tfrac{1}{3} m^3 (1 + \beta) \\
1^3 + 2^3 + 3^3 + \ldots\ldots + m^3 & \ldots \tfrac{1}{4} m^4 (1 + \gamma) \quad \text{etc.}
\end{array}$$

where α, β, γ, etc., diminish without limit, when m is increased without limit. If we substitute these values, and also put $\frac{h}{m}$ instead of dx, we have, for the sum of the terms,

$$\varphi a\, h + \varphi' a\, \frac{h^2}{2}(1+\alpha) + \varphi'' a\, \frac{h^3}{2.3}(1+\beta)$$
$$+ \varphi''' a\, \frac{h^4}{2.3.4}(1+\gamma) + \text{etc.}$$

which, when m is increased without limit, in consequence of which α, β, etc., diminish without limit, continually approaches to

$$\varphi a\, h + \varphi' a\, \frac{h^2}{2} + \varphi'' a\, \frac{h^3}{2.3} + \varphi''' a\, \frac{h^4}{2.3.4} + \text{etc.}$$

which is the limit arising from supposing x to increase from a through $a+dx$, $a+2dx$, etc., up to $a+h$, multiplying every value of φx so obtained by dx, summing the results, and decreasing dx without limit.

This is the integral of $\varphi x\, dx$ from $x=a$ to $x= a+h$. It is evident that this series bears a great resemblance to the development in page 21, deprived of its first term. Let us suppose that ψa is the function of which φa is the differential coefficient, that is, that $\psi' a = \varphi a$. These two functions being the same, their differential coefficients will be the same, that is, $\psi'' a = \varphi' a$. Similarly $\psi''' a = \varphi'' a$, and so on. Substituting these, the above series becomes

$$\psi' a\, h + \psi'' a\, \frac{h^2}{2} + \psi''' a\, \frac{h^3}{2.3} + \psi^{\text{IV}} a\, \frac{h^4}{2.3.4} + \text{etc.}$$

which is (page 21) the same as $\psi(a+h) - \psi a$. That is, the integral of $\varphi x\, dx$ between the limits a and $a+h$, is $\psi(a+h) - \psi a$, where ψx is the function, which,

when differentiated, gives φx. For $a+h$ we may write b, so that $\psi b - \psi a$ is the integral of $\varphi x\, dx$ from $x=a$ to $x=b$. Or we may make the second limit indefinite by writing x instead of b, which gives $\psi x - \psi a$, which is said to be the integral of $\varphi x\, dx$, beginning when $x=a$, the summation being supposed to be continued from $x=a$ until x has the value which it may be convenient to give it.

NATURE OF INTEGRATION.

Hence results a new branch of the inquiry, the reverse of the Differential Calculus, the object of which is, not to find the differential coefficient, having given the function, but to find the function, having given the differential coefficient. This is called the Integral Calculus.

From the definition given, it is obvious that the value of an integral is not to be determined, unless we know the values of x corresponding to the beginning and end of the summation, whose limit furnishes the integral. We might, instead of defining the integral in the manner above stated, have made the word mean merely the converse of the differential coefficient; thus, if φx be the differential coefficient of ψx, ψx might have been called the integral of $\varphi x\, dx$. We should then have had to show that the integral, thus defined, is equivalent to the limit of the summation already explained. We have preferred bringing the former method before the student first, as it is most analogous to the manner in which he will deduce integrals in questions of geometry or mechanics.

With the last-mentioned definition, it is also obvious that every function has an unlimited number of integrals. For whatever differential coefficient ψx

gives, $C+\psi x$ will give the same, if C be a constant, that is, not varying when x varies. In this case, if x become $x+h$, $C+\psi x$ becomes $C+\psi x+\psi' x h+$ etc., from which the subtraction of the original form $C+\psi x$ gives $\psi' x h+$ etc.; whence, by the process in page 23, $\psi' x$ is the differential coefficient of $C+\psi x$ as well as of ψx. As many values, therefore, positive or negative, as can be given to C, so many different integrals can be found for $\psi' x$; and these answer to the various limits between which the summation in our original definition may be made. To make this problem definite, not only $\psi' x$ the function to be integrated, must be given, but also that value of x from which the summation is to begin. If this be a, the integral of $\psi' x$ is, as before determined, $\psi x-\psi a$, and $C=-\psi a$. We may afterwards end at any value of x which we please. If $x=a$, $\psi x-\psi a=0$, as is evident also from the formation of the integral. We may thus, having given an integral in terms of x, find the value at which it began, by equating the integral to zero, and finding the value of x. Thus, since x^2, when differentiated, gives $2x$, x^2 is the integral of $2x$, beginning at $x=0$; and x^2-4 is the integral beginning at $x=2$.

In the language of Leibnitz, an integral would be the sum of an infinite number of infinitely small quantities, which are the differentials or infinitely small increments of a function. Thus, a circle being, according to him, a rectilinear polygon of an infinite number of infinitely small sides, the sum of these would be the circumference of the figure. As before (pages 13–14, 38 et seq., 48 et seq.) we proceed to interpret this inaccuracy of language. If, in a circle, we successively describe regular polygons of 3, 4, 5, 6, etc., sides, we may, by this means, at last attain to a poly-

gon whose side shall differ from the arc of which it is the chord, by as small a fraction, either of the chord or arc, as we please (pages 7–11). That is, A being the arc, C the chord, and D their difference, there is no fraction so small that D cannot be made a smaller part of C. Hence, if m be the number of sides of the polygon, $m\mathrm{C} + m\mathrm{D}$ or $m\mathrm{A}$ is the real circumference; and since $m\mathrm{D}$ is the same part of $m\mathrm{C}$ which D is of C, $m\mathrm{D}$ may be made as small a part of $m\mathrm{C}$ as we please; so that $m\mathrm{C}$, or the sum of all the sides of the polygon, can be made as nearly equal to the circumference as we please.

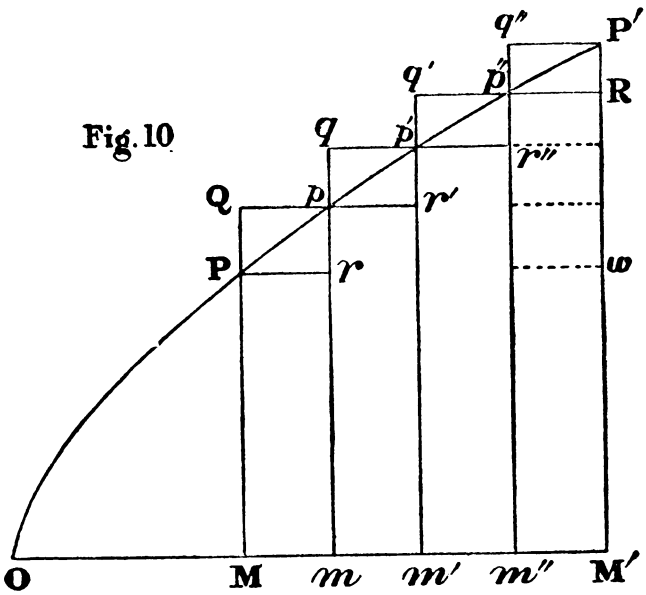
As in other cases, the expressions of Leibnitz are the most convenient and the shortest, for all who can immediately put a rational construction upon them; this, and the fact that, good or bad, they have been, and are, used in the works of Lagrange, Laplace, Euler, and many others, which the student who really desires to know the present state of physical science, cannot dispense with, must be our excuse for continually bringing before him modes of speech, which, taken quite literally, are absurd.

DETERMINATION OF CURVILINEAR AREAS. THE PARABOLA.

We will now suppose such a part of a curve, each ordinate of which is a given function of the corresponding abscissa, as lies between two given ordinates; for example, MPP′M′. Divide the line MM′ into a number of equal parts, which we may suppose as great as we please, and construct Figure 10. Let O be the origin of co-ordinates, and let OM, the value of x, at which we begin, be a; and OM′, the value at which we end, be b. Though we have only divided

MM′ into four equal parts in the figure, the reasoning to which we proceed would apply equally, had we divided it into four million of parts. The sum of the parallelograms Mr, mr', $m'r''$, and m''R, is less than the area MPP′M′, the value of which it is our object to investigate, by the sum of the curvilinear triangles Prp, $pr'p'$, $p'r''p''$, and p''RP′. The sum of these triangles is less than the sum of the parallelograms Qr, qr', $q'r''$, and q''R; but these parallelograms are to-

Fig. 10



gether equal to the parallelogram $q''w$, as appears by inspection of the figure, since the base of each of the above-mentioned parallelograms is equal to m''M′, or q''P′, and the altitude P′w is equal to the sum of the altitudes of the same parallelograms. Hence the sum of the parallelograms Mr, mr' $m'r''$, and m''R, differs from the curvilinear area MPP′M′ by less than the parallelogram $q''w$. But this last parallelogram may be made as small as we please by sufficiently increasing the number of parts into which MM′ is divided;

for since one side of it, $P'w$, is always less than $P'M'$, and the other side $P'q''$, or $m''M'$, is as small a part as we please of MM' the number of square units in $q''w$, is the product of the number of linear units in $P'w$ and $P'q''$, the first of which numbers being finite, and the second as small as we please, the product is as small as we please. Hence the curvilinear area $MPP'M'$ is the limit towards which we continually approach, but which we never reach, by dividing MM' into a greater and greater number of equal parts, and adding the parallelograms Mr, mr', etc., so obtained. If each of the equal parts into which MM' is divided be called dx, we have $OM = a$, $Om = a + dx$, $Om' = a + 2dx$, etc. And MP, mp, $m'p'$, etc., are the values of the function which expresses the ordinates, corresponding to a, $a + dx$, $a + 2dx$, etc., and may therefore be represented by φa, $\varphi(a + dx)$, $\varphi(a + 2dx)$, etc. These are the altitudes of a set of parallelograms, the base of each of which is dx; hence the sum of their area is

$$\varphi a\, dx + \varphi(a + dx)\, dx + \varphi(a + 2dx)\, dx + \text{ etc.},$$

and the limit of this, to which we approach by diminishing dx, is the area required.

This limit is what we have defined to be the integral of $\varphi x\, dx$ from $x = a$ to $x = b$; or if ψx be the function, which, when differentiated, gives φx, it is $\psi b - \psi a$. Hence, y being the ordinate, the area included between the axis of x, any two values of y, and the portion of the curve they cut off, is $\int y dx$, beginning at the one ordinate and ending at the other.

Suppose that the curve is a part of a parabola of which O is the vertex, and whose equation* is

* If the student has not any acquaintance with the conic sections, he must nevertheless be aware that there is some curve whose abscissa and ordinate

therefore $y^2 = px$ where p is the double ordinate which passes through the focus. Here $y = p^{\frac{1}{2}} x^{\frac{1}{2}}$, and we must find the integral of $p^{\frac{1}{2}} x^{\frac{1}{2}} dx$, or the function whose differential coefficient is $p^{\frac{1}{2}} x^{\frac{1}{2}}$, $p^{\frac{1}{2}}$ being a constant. If we take the function cx^n, c being independent of x, and substitute $x + h$ for x, we have for the development $cx^n + cnx^{n-1}h +$ etc. Hence the differential coefficient of cx^n is cnx^{n-1}; and as c and n may be any numbers or fractions we please, we may take them such that cn shall $= p^{\frac{1}{2}}$ and $n - 1 = \frac{1}{2}$, in which case $n = \frac{3}{2}$ and $c = \frac{2}{3} p^{\frac{1}{2}}$. Therefore the differential coefficient of $\frac{2}{3} p^{\frac{1}{2}} x^{\frac{3}{2}}$ is $p^{\frac{1}{2}} x^{\frac{1}{2}}$, and conversely, the integral of $p^{\frac{1}{2}} x^{\frac{1}{2}} dx$ is $\frac{2}{3} p^{\frac{1}{2}} x^{\frac{3}{2}}$.

The area MPP′M′ of the parabola is therefore $\frac{2}{3} p^{\frac{1}{2}} b^{\frac{3}{2}} - \frac{2}{3} p^{\frac{1}{2}} a^{\frac{3}{2}}$. If we begin the integral at the vertex O, in which case $a = 0$, we have for the area OM′P′, $\frac{2}{3} p^{\frac{1}{2}} b^{\frac{3}{2}}$, where $b =$ OM′. This is $\frac{2}{3} p^{\frac{1}{2}} b^{\frac{1}{2}} \times b$, which, since $p^{\frac{1}{2}} b^{\frac{1}{2}} =$ M′P′ is $\frac{2}{3}$P′M′ $\times$ OM′, or two-thirds of the rectangle* contained by OM′ and M′P′.

METHOD OF INDIVISIBLES.

We may mention, in illustration of the preceding problem, a method of establishing the principles of the Integral Calculus, which generally goes by the name of the *Method of Indivisibles.* A line is considered as the sum of an infinite number of points, a surface of an infinite number of lines, and a solid of an infinite number of surfaces. One line twice as long as another would be said to contain twice as many

are connected by the equation $y^2 = px$. This, to him, must be the definition of *parabola;* by which word he must understand, a curve whose equation is $y^2 = px$.

*This proposition is famous as having been discovered by Archimedes at a time when such a step was one of no small magnitude.

points, though the number of points in each is unlimited. To this there are two objections. First, the word infinite, in this absolute sense, really has no meaning, since it will be admitted that the mind has no conception of a number greater than any number. The word infinite* can only be justifiably used as an abbreviation of a distinct and intelligible proposition; for example, when we say that $a+\frac{1}{x}$ is equal to a when x is infinite, we only mean that as x is increased, $a+\frac{1}{x}$ becomes nearer to a, and may be made as near to it as we please, if x may be as great as we please. The second objection is, that the notion of a line being the sum of a number of points is not true, nor does it approach nearer the truth as we increase the number of points. If twenty points be taken on a straight line, the sum of the twenty-one lines which lie between point and point is equal to the whole line; which cannot be if the points by themselves constitute any part of the line, however small. Nor will the sum of the points be a part of the line, if twenty thousand be taken instead of twenty. There is then, in this method, neither the rigor of geometry, nor that approach to truth, which, in the method of Leibnitz, may be carried to any extent we please, short of absolute correctness. We would therefore recommend to the student not to regard any proposition derived from this method as true on that account; for falsehoods, as well as truths, may be deduced from it. Indeed, the primary notion, that the number of points in a line is proportional to its length, is manifestly incorrect. Suppose (Fig. 6, page 48) that the point Q

* See *Study of Mathematics* (Chicago: The Open Court Publishing Co.), page 123 et seq.

moves from A to P. It is evident that in whatever number of points OQ cuts AP, it cuts MP in the same number. But PM and PA are not equal. A defender of the system of indivisibles, if there were such a person, would say something equivalent to supposing that the points on the two lines are of *different sizes*, which would, in fact, be an abandonment of the method, and an adoption of the idea of Leibnitz, using the word *point* to stand for the infinitely small line.

This notion of indivisibles, or at least a way of speaking which looks like it, prevails in many works on mechanics. Though a point is not treated as a length, or as any part of space whatever, it is considered as having weight; and two points are spoken of as having different weights. The same is said of a line and a surface, neither of which can correctly be supposed to possess weight. If a solid be of the same density throughout, that is, if the weight of a cubic inch of it be the same from whatever part it is cut, it is plain that the weight may be found by finding the number of cubic inches in the whole, and multiplying this number by the weight of one cubic inch. But if the weight of every two cubic inches is different, we can only find the weight of the whole by the integral calculus.

Let AB (Fig. 11) be a line possessing weight, or a very thin parallelepiped of matter, which is such, that if we were to divide it into any number of equal parts, as in the figure, the weight of the several parts would be different. We suppose the weight to vary continuously, that is, if two contiguous parts of equal length be taken, as pq and qr, the ratio of the weights

of these two parts may, by taking them sufficiently small, be as near to equality as we please.

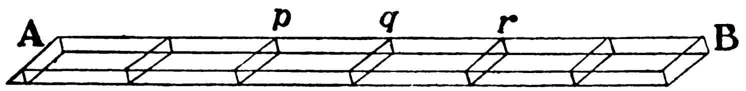
The *density* of a body is a mathematical term, which may be explained as follows: A cubic inch of gold weighs more than a cubic inch of water; hence gold is *denser* than water. If the first weighs 19 times as much as the second, gold is said to be 19 times more dense than water, or the density of gold is 19 times that of water. Hence we might define the density by the weight of a cubic inch of the substance, but it is usual to take, not this weight, but the proportion which it bears to the same weight of water. Thus, when we say the *density*, or *specific gravity* (these terms are used indifferently), of cast iron is 7·207, we mean that if any vessel of pure water were emptied and filled with cast iron, the iron would weigh 7·207 times as much as the water.

If the density of a body were uniform throughout, we might easily determine it by dividing the weight of any bulk of the body, by the weight of an equal bulk of water. In the same manner (pages 52 et seq.) we could, from our definition of velocity, determine any uniform velocity by dividing the length described by the time. But if the density vary continuously, no such measure can be adopted. For if by the side of AB (which we will suppose to be of iron) we placed a similar body of water similarly divided, and if we divided the weight of the part *pq* of iron by the weight of the same part of water, we should get different densities, according as the part *pq* is longer or shorter. The water is supposed to be homogeneous, that is, any part of it *pr*, being twice the length of *pq*, is twice the weight of *pq*, and so on. The iron, on the contrary, being supposed to vary in density, the doubling

the length gives either more or less than twice the weight. But if we suppose q to move towards p, both on the iron and the water, the limit of the ratio pq of iron to pq of water, may be chosen as a measure of the density of p, on the same principle as in pages 54–55, the limit of the ratio of the length described to the time of describing it, was called the velocity. If we call k this limit, and if the weight varies continuously, though no part pq, however small, of iron, would be exactly k times the same part of water in weight, we may nevertheless take pq so small that these weights shall be as nearly as we please in the ratio of k to 1.

Let us now suppose that this density, expressed by the limiting ratio aforesaid, is always x^2 at any

Fig. 11.



point whose distance from A is x feet; that is, the density at q, 2 feet distance from A, is 4, and so on. Let the whole distance $AB = a$. If we divide a into n equal parts, each of which is dx, so that $ndx = a$, and if we call b the area of the section of the parallelepiped, (b being a fraction of a square foot,) the solid content of each of the parts will be bdx in cubic feet; and if w be the weight of a cubic foot of water, the weight of the same bulk of water will be $wbdx$. If the solid AB were homogeneous in the immediate neighborhood of the point p, the density being then x^2, would give $x^2 \times bwdx$ for the weight of the same part of the substance. This is not true, but can be brought as near to the truth as we please, by taking dx sufficiently small, or dividing AB into a suffi-

cient number of parts. Hence the real weight of pq may be represented by $bwx^2dx + \alpha$, where α may be made as small a part as we please of the term which precedes it.

In the sum of any number of these terms, the sum arising from the term α diminishes without limit as compared with the sum arising from the term bwx^2dx; for if α be less than the thousandth part of p, α' less than the thousandth part of p', etc., then $\alpha + \alpha' +$ etc. will be less than the thousandth part of $p + p' +$ etc.: which is also true of any number of quantities, and of any fraction, however small, which each term of one set is of its corresponding term in the other. Hence the taking of the integral of bwx^2dx dispenses with the necessity of considering the term α; for in taking the integral, we find a limit which supposes dx to have decreased without limit, and the *integral* which would arise from α has therefore diminished without limit.

The integral of bwx^2dx is $\frac{1}{3}bwx^3$, which taken from $x=0$ to $x=a$ is $\frac{1}{3}bwa^3$. This is therefore the weight in pounds of the bar whose length is a feet, and whose section is b square feet, when the density at any point distant by x feet from the beginning is x^2; w being the weight in pounds of a cubic foot of water.

CONCLUDING REMARKS ON THE STUDY OF THE CALCULUS.

We would recommend it to the student, in pursuing any problem of the Integral Calculus, never for one moment to lose sight of the manner in which he would do it, if a rough solution for practical purposes only were required. Thus, if he has the area of a curve to find, instead of merely saying that y, the ordinate, being a certain function of the abscissa x,

$\int ydx$ within the given limits would be the area required; and then proceeding to the mechanical solution of the question: let him remark that if an approximate solution only were required, it might be obtained by dividing the curvilinear area into a number of four-sided figures, as in Figure 10, one side of which only is curvilinear, and embracing so small an arc that it may, without visible error, be considered as rectilinear. The mathematical method begins with the same principle, investigating upon this supposition, not the sum of these rectilinear areas, but the limit towards which this sum approaches, as the subdivision is rendered more minute. This limit is shown to be that of which we are in search, since it is proved that the error diminishes without limit, as the subdivision is indefinitely continued.

We now leave our reader to any elementary work which may fall in his way, having done our best to place before him those considerations, something equivalent to which he must turn over in his mind before he can understand the subject. The method so generally followed in our elementary works, of leading the student at once into the mechanical processes of the science, postponing entirely all other considerations, is to many students a source of obscurity at least, if not an absolute impediment to their progress; since they cannot imagine what is the object of that which they are required to do. That they shall understand everything contained in these treatises, on the first or second reading, we cannot promise; but that the want of illustration and the preponderance of *technical* reasoning are the great causes of the difficulties which students experience, is the opinion of many who have had experience in teaching this subject.

BRIEF BIBLIOGRAPHY.*

STANDARD TEXT-BOOKS AND TREATISES ON THE CALCULUS.

ENGLISH.

Perry, John: *Calculus for Engineers.* Second edition, London and New York: Edward Arnold. 1897. Price, 7s. 6d. ($2.50).

Extract from Author's Preface: "This book describes what has for many years been the most important part of the regular course in the Calculus for Mechanical and Electrical Engineering students at the Finsbury Technical College. The students in October knew only the most elementary mathematics, many of them did not know the Binomial Theorem, or the definition of the sine of an angle. In July they had not only done the work of this book, but their knowledge was of a practical kind, ready for use in any such engineering problems as I give here."

Especially good in the character and number of practical examples given.

Lamb, Horace: *Infinitesimal Calculus.* New York: The Macmillan Co. 1898. Price, $3.00.

Extract from Author's Preface: "This book attempts to teach those portions of the Calculus which are of primary importance in the application to such subjects as Physics and Engineering.... Stress is laid on fundamental principles.... Considerable attention has been paid to the logic of the subject."

*The information given regarding the works mentioned in this list is designed to enable the reader to select the books which are best suited to his needs and his purse. Where the titles do not sufficiently indicate the character of the books, a note or extract from the Preface has been added. The American prices have been supplied by Messrs. Lemcke & Buechner, 812 Broadway, New York, through whom the purchases, especially of the foreign books, may be conveniently made.—*Ed.*

Edwards, Joseph: *An Elementary Treatise on the Differential Calculus*. Second edition, revised. 8vo, cloth. New York and London: The Macmillan Co. 1892. Price, $3.50.—*Differential Calculus for Beginners*. 8vo, cloth. 1893. *The Integral Calculus for Beginners*. 8vo, cloth. (Same Publishers.) Price, $1.10 each.

Byerly, William E.: *Elements of the Differential Calculus*. Boston: Ginn & Co. Price, $2.15.—*Elements of the Integral Calculus*. (Same Publishers.) Price, $2.15.

Rice, J. M., and Johnson, W. W.: *An Elementary Treatise on the Differential Calculus Founded on the Method of Rates or Fluxions*. New York: John Wiley & Sons. 8vo. 1884. Price, $3.50. Abridged edition, 1889. Price, $1.50.

Johnson, W. W.: *Elementary Treatise on the Integral Calculus Founded on the Method of Rates or Fluxions*. 8vo, cloth. New York: John Wiley & Sons. 1885. Price, $1.50.

Greenhill, A. G.: *Differential and Integral Calculus*. With applications. 8vo, cloth. Second edition. New York and London: The Macmillan Co. 1891. Price, 9s. ($2.60).

Price: *Infinitesimal Calculus*. Four Vols. 1857–65. Out of print and very scarce. Obtainable for about $27.00.

Smith, William Benjamin: *Infinitesimal Analysis*. Vol. I., Elementary: Real Variables. New York and London: The Macmillan Co. 1898. Price, $3.25.

"The aim has been, within a prescribed expense of time and energy to penetrate as far as possible, and in as many directions, into the subject in hand,—that the student should attain as wide knowledge of the matter, as full comprehension of the methods, and as clear consciousness of the spirit and power of analysis as the nature of the case would admit."—From Author's Preface.

Todhunter, Isaac: *A Treatise on the Differential Calculus*. London and New York: The Macmillan Co. Price, 10s. 6d. ($2.60). *A Treatise on the Integral Calculus*. (Same publishers.) Price, 10s. 6d. ($2.60).

Todhunter's text-books were, until recently, the most widely used in England. His works on the Calculus still retain their standard character, as general manuals.

Williamson: *Differential and Integral Calculus*. London and New York: Longmans, Green, & Co. 1872–1874. Two Vols. Price, $3.50 each.

De Morgan, Augustus: *Differential and Integral Calculus*. London: Society for the Diffusion of Useful Knowledge. 1842. Out of print. About $6.40.

The most extensive and complete work in English. "The object has been to contain within the prescribed limits, the whole of the students' course from the confines of elementary algebra and trigonometry, to the entrance of the highest works on mathematical physics" (Author's Preface). Few examples. In typography, and general arrangement of material, inferior to the best recent works. Valuable for collateral study, and for its philosophical spirit.

FRENCH.

Sturm: *Cours d'analyse de l'École Polytechnique*. 10. édition, revue et corrigé par E. Prouhet, et augmentée de la théorie élémentaire des fonctions elliptiques, par H. Laurent. 2 volumes in—8. Paris: Gauthier-Villars et fils. 1895. Bound, 16 fr. 50 c. $4.95.

One of the most widely used of text-books. First published in 1857. The new tenth edition has been thoroughly revised and brought down to date. The exercises, while not numerous, are sufficient, those which accompany the additions and complementary chapters of M. De Saint Germain having been taken from the Collection of M. Tisserand, mentioned below.

Duhamel: *Éléments de calcul infinitésimal*. 4. édition, revue et annotée par J. Bertrand. 2 volumes in—8; avec planches. Paris: Gauthier-Villars et fils. 1886. 15 fr. $4.50.

The first edition was published between 1840 and 1841. "Cordially recommended to teachers and students" by De Morgan. Duhamel paid great attention to the philosophy and logic of the mathematical sciences, and the student may also be referred in this connexion to his *Méthodes dans les sciences de raisonnement*. 5 volumes. Paris: Gauthier-Villars et fils. Price, 25.50 francs. $7.65.

Lacroix, S.-F.: *Traité élémentaire de calcul différentiel et de calcul intégral*. 9. édition, revue et augmentée de notes par Hermite et Serret. 2 vols. Paris: Gauthier-Villars et fils. 1881. 15 fr. $4.50.

A very old work. The first edition was published in 1797. It was the standard treatise during the early part of the century, and has been kept revised by competent hands.

Appell, P.: *Éléments d'analyse mathématique.* À l'usage des ingénieurs et des physiciens. Cours professé à l'École Centrale des Arts et Manufactures. 1 vol. in—8, 720 pages, avec figures, cartonné à l'anglaise. Paris: Georges Carré & C. Naud. 1899. Price, 24 francs. $7.20.

Boussinesq, J.: *Cours d'analyse infinitésimal.* À l'usage des personnes qui étudient cette science en vue de ses applications mécaniques et physiques, 2 vols., grand in-8, avec figures. Tome I. Calcul différentiel. Paris, 1887. 17 fr. ($5.10). Tome II. Calcul intégral. Paris: Gauthier-Villars et fils. 1890. 23 fr. 50 c. ($7.05).

Hermite, Ch.: *Cours d'analyse de l'École Polytechnique.* 2 vols. Vol. I. Paris: Gauthier-Villars et fils. 1897.

A new edition of Vol. I. is in preparation (1899). Vol. II. has not yet appeared.

Jordan, Camille: *Cours d'analyse de l'École Polytechnique.* 3 volumes. 2. édition. Paris: Gauthier-Villars et fils. 1893–1898. 51 fr. $14.70.

Very comprehensive on the theoretical side. Enters deeply into the metaphysical aspects of the subject.

Laurent, H.: *Traité d'analyse.* 7 vols in—8. Paris: Gauthier-Villars et fils. 1885–1891. 73 fr. $21.90.

The most extensive existing treatise on the Calculus. A general handbook and work of reference for the results contained in the more special works and memoirs.

Picard, Émile: *Traité d'analyse.* 4 volumes grand in-8. Paris: Gauthier-Villars et fils. 1891. 15 fr. each. Vols. I.—III., $14.40. Vol. IV. has not yet appeared.

An advanced treatise on the Integral Calculus and the theory of differential equations. Presupposes a knowledge of the Differential Calculus.

Serret, J.-A.: *Cours de calcul différentiel et intégral.* 4. edition, augmentée d'une note sur les fonctions elliptiques, par Ch. Hermite. 2 forts volumes in—8. Paris: Gauthier-Villars et fils. 1894. 25 fr. $7.50.

A good German translation of this work by Axel Harnack has passed through its second edition (Leipsic: Teubner, 1885 and 1897).

Houël, J.: *Cours de calcul infinitésimal.* 4 beaux volumes grand in—8, avec figures. Paris: Gauthier-Villars et fils. 1878–1879–1880–1881. 50 fr. $15.00.

Bertrand, J.: *Traité de calcul différentiel et de calcul intégral.* (1) Calcul différentiel. Paris: Gauthier-Villars et fils. 1864. Scarce. About $48.00. (2) Calcul intégral (Intégrales définies et indéfinies). Paris, 1870. Scarce. About $24.00.

Boucharlat, J.-L.: *Éléments de calcul différentiel et dė calcul intégral.* 9. édition, revue et annotée par H. Laurent. Paris: Gauthier-Villars et fils. 1891. 8 fr. $2.40.

Moigno: *Leçons de calcul différentiel et de calcul intégral,* 2 vols., Paris, 1840–1844. Scarce. About $9.60.

Navier: *Leçons d'analyse de l'École Polytechnique.* Paris, 1840. 2nd ed. 1856. Out of print. About $3.60.

An able and practical work. Very popular in its day. The typical course of the *Ecole Polytechnique*, and the basis of several of the treatises that followed, including that of Sturm. Also much used in its German translation.

Cournot: *Théorie des fonctions et du calcul infinitésimal.* 2 vols. Paris, 1841. 2nd ed. 1856–1858. Out of print, and scarce. About $3.00.

The first edition (1841) was "cordially recommended to teachers and students" by De Morgan. Cournot was especially strong on the philosophical side. He examined the foundations of many sciences and developed original views on the theory of knowledge, which are little known but have been largely drawn from by other philosophers.

Cauchy, A.: *Œuvres complètes.* Tome III: *Cours d'analyse de l'École Polytechnique.* Tome IV: *Résumé des leçons données à l'École Polytechnique sur le calcul infinitésimal. Leçons sur le calcul différentiel.* Tome V: *Leçons sur les applications du calcul infinitésimal à la géométrie.* Paris: Gauthier-Villars et fils, 1885–1897. 25 fr. each. $9.50 each.

The works of Cauchy, as well as those of Lagrange, which follow, are mentioned for their high historical and educational importance.

Lagrange, J. L.: *Œuvres complètes.* Tome IX: *Théorie des fonctions analytiques.* Tome X.: *Leçons sur le calcul des fonc-*

tions. Paris: Gauthier-Villars et fils, 1881–1884. 18 fr. per volume. $5.40 per volume.

"The same power of abstraction and facility of treatment which signalise these works are nowhere to be met with in the prior or subsequent history of the subject. In addition, they are replete with the profoundest aperçus into the history of the development of analytical truths,—aperçus which could have come only from a man who combined superior creative endowment with exact and comprehensive knowledge of the facts. In the remarks woven into the body of the text will be found what is virtually a detailed history of the subject, and one which is not to be had elsewhere, least of all in diffuse histories of mathematics. The student, thus, not only learns in these works how to think, but also discovers how people actually have thought, and what are the ways which human instinct and reason have pursued in the different individuals who have participated in the elaboration of the science."—(E. Dühring.)

Euler, L.:

The Latin treatises of Euler are also to be mentioned in this connexion, for the benefit of those who wish to pursue the history of the text-book making of this subject to its fountain-head. They are the *Differential Calculus* (St. Petersburg, 1755), the *Integral Calculus* (3 vols., St. Petersburg, 1768–1770), and the *Introduction to the Infinitesimal Analysis* (2 vols., Lausanne, 1748). Of the last-mentioned work an old French translation by Labey exists (Paris: Gauthier-Villars), and a new German translation (of Vol. I. only) by Maser (Berlin: Julius Springer, 1885). Of the first-mentioned treatises on the Calculus proper there exist two old German translations, which are not difficult to obtain.

GERMAN.

Harnack, Dr. Axel: *Elemente der Differential- und Integralrechnung.* Zur Einführung in das Studium dargestellt. Leipzig: Teubner, 1881. M. 7.60. Bound, $2.80. (English translation. London: Williams & Norgate. 1891.)

Junker, Dr. Friedrich: *Höhere Analysis.* I. *Differentialrechnung.* Mit 63 Figuren. II. *Integralrechnung.* Leipzig: G. J. Göschen'sche Verlagshandlung. 1898–1899. 80 pf. each. 30 cents each.

These books are marvellously cheap, and very concise. They contain no examples. Pocket-size.

Autenheimer, F.: *Elementarbuch der Differential- und Integralrechnung mit zahlreichen Anwendungen aus der Analysis, Geometrie, Mechanik, Physik etc.* Für höhere Lehranstalten

und den Selbstunterricht. 4te verbesserte Auflage. Weimar: Bernhard Friedrich Voigt. 1895.

As indicated by its title, this book is specially rich in practical applications.

Stegemann: *Grundriss der Differential- und Integralrechnung*, 8te Auflage, herausgegeben von Kiepert. Hannover: Helwing, 1897. Two volumes, 26 marks. Two volumes, bound, $8.50.

This work was highly recommended by Prof. Felix Klein at the Evanston Colloquium in 1893.

Schlömilch: *Compendium der höheren Analysis.* Fifth edition, 1881. Two volumes, $6.80.

Schlömilch's text-books have been very successful. The present work was long the standard manual.

Stolz, Dr. Otto: *Grundzüge der Differential- und Integralrechnung.* In 2 Theilen. I. Theil. Reelle Veränderliche und Functionen. (460 S.) 1893. M. 8. II. Complexe Veränderliche und Functionen. (338 S.) Leipzig: Teubner. 1896. M. 8. Two volumes, $6.00.

A supplementary 3rd part entitled *Die Lehre von den Doppelintegralen* has just been published (1899). Based on the works of J. Tannery, Peano, and Dini.

Lipschitz, R.: *Lehrbuch der Analysis.* 1877–1880. Two volumes, bound, $12.30.

Specially good on the theoretical side.

COLLECTIONS OF EXAMPLES AND ILLUSTRATIONS.

Byerly, W. E.: *Problems in Differential Calculus.* Supplementary to a Treatise on Differential Calculus. Boston: Ginn & Co. 75 cents.

Gregory: *Examples on the Differential and Integral Calculus.* 1841. Second edition. 1846. Out of print. About $6.40.

Frenet: *Recueil d'exercises sur le calcul infinitésimal.* 5. édition, augmentée d'un appendice, par H. Laurent. Paris: Gauthier-Villars et fils. 1891. 8 fr. $2.40.

Tisserand, F.: *Recueil complémentaire d'exercises sur le calcul infinitésimal.* Second edition. Paris: Gauthier-Villars et fils. 1896.

Complementary to Frenet.

Laisant, C. A.: *Recueil de problèmes de mathématiques.* Tome VII. Calcul infinitésimal et calcul des fonctions. Mécanique. Astronomie. (Announced for publication.) Paris: Gauthier-Villars et fils.

Schlömilch, Dr. Oscar: *Uebungsbuch zum Studium der höheren Analysis.* I. Theil. Aufgaben aus der Differentialrechnung. 4te Auflage. (336 S.) 1887. M. 6. II. Aufgaben aus der Integralrechnung. 3te Auflage. (384 S.) Leipzig: Teubner, 1882. M. 7.60. Both volumes, bound, $7.60.

Sohncke, L. A.: *Sammlung von Aufgaben aus der Differential- und Integralrechnung.* Herausgegeben von Heis. Two volumes. in—8. Bound, $3.00.

Fuhrmann, Dr. Arwed: *Anwendungen der Infinitesimalrechnung in den Naturwissenschaften, im Hochbau und in der Technik.* Lehrbuch und Aufgabensammlung. In sechs Theilen, von denen jeder ein selbstständiges Ganzes bildet. Theil I. Naturwissenschaftliche Anwendungen der Differentialrechnung. Theil II. Naturwissenschaftliche Anwendungen der Integralrechnung. Berlin: Verlag von Ernst & Korn. 1888–1890. Vol. I., Cloth, $1.35. Vol. II., Cloth, $2.20.

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THE FOUNDATIONS OF GEOMETRY, by David Hilbert, Ph. D., Professor of Mathematics, University of Göttingen. Authorized translation by Prof. E. J. Townsend, Ph. D., University of Illinois. 1902. Pp. vii., 143. Cloth $1.00 net. (4s. 6d. net.)

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MATHEMATICAL ESSAYS AND RECREATIONS, by Hermann Schubert, Professor of Mathematics in the Johanneum. Hamburg, Germany. From the German, by Thomas J. McCormack. Second edition. 1903. 37 cuts. Pp. 149. Cloth 75c net. (3s. 6d. net.)

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GEOMETRIC EXERCISES IN PAPER FOLDING, by T. Sundara Row. Edited and revised by Wooster Woodruff Beman, Professor of Mathematics in the University of Michigan, and David Eugene Smith, Professor of Mathematics in Teachers' College of Columbia University. With 87 illustrations. 1901. Pp. xiv., 148. Cloth $1.00 net. (4s. 6d. net.)

A SCRAPBOOK OF ELEMENTARY MATHEMATICS, NOTES, RECREATIONS, ESSAYS, by William F. White, Ph. D., State Normal School, New Paltz, New York. Frontispiece and 70 diagrams and other illustrations. 12 mo. 248 pp., cloth binding, gilt top, price $1.00 net. (5s. net.)